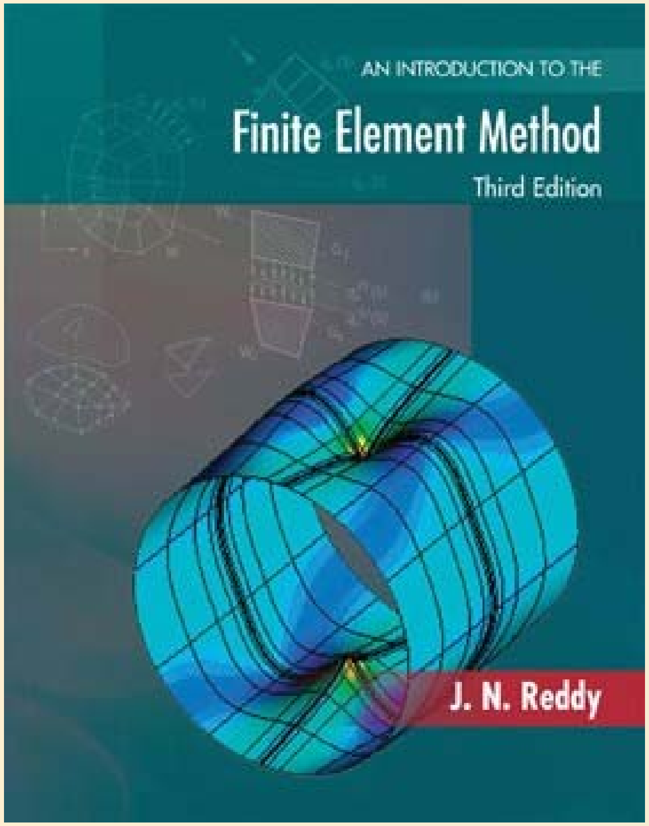
The Finite Element Method

1D Problems Governed by Second-Order Equation

Read: Chs. 3 and 4



CONTENTS

- Model differential equation
- Finite element approximation
- Finite element discretization
- Development of weak form and the definition of primary and secondary variables (duality)
- Essential and natural BCs
- Linear and bilinear forms
- Finite element model
- Interpolation functions
- Assembly of elements
- Numerical examples

Important Comments

- **Knowing is power. Your confidence in what you do goes up.**
- **Mathematics is the language of an engineer.**
- **"Mathematics is the language with which God has written the universe." –*Galileo Galilei***

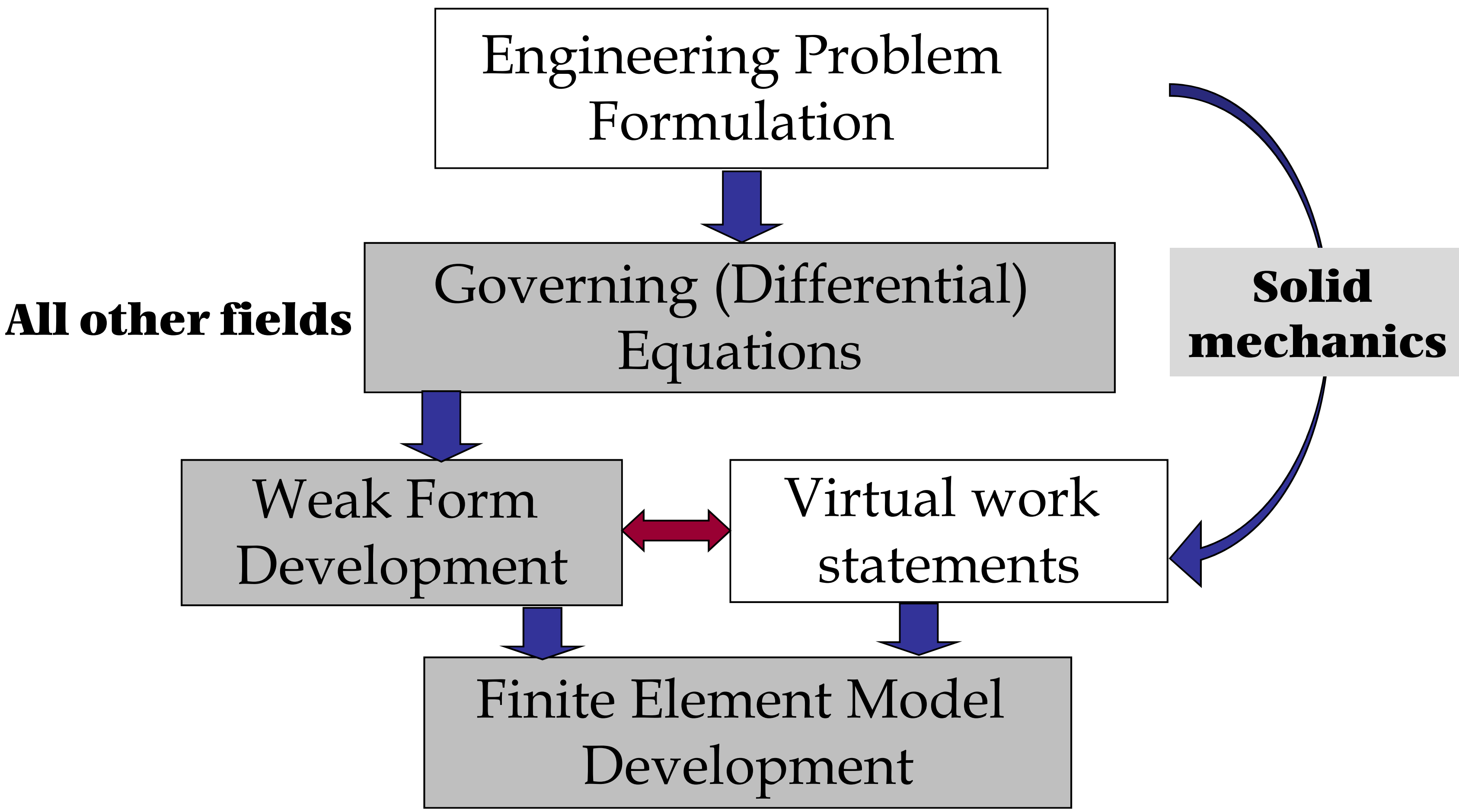
When you are analyzing an engineering problem using a FEM program, you should know:

(1) the mathematical model (governing equations) that the program is solving,

(2) the duality pairs for specifying the boundary conditions, and

(3) know the restrictions of the mathematical and computational models.

Major Steps of Finite Element Formulation



Model Differential Equation Boundary Conditions and data

Governing equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$

Boundary conditions

$$u=\hat{u} \quad \text{or} \quad a\frac{du}{dx}+b(u-u_0)=P \text{ at a boundary point}$$

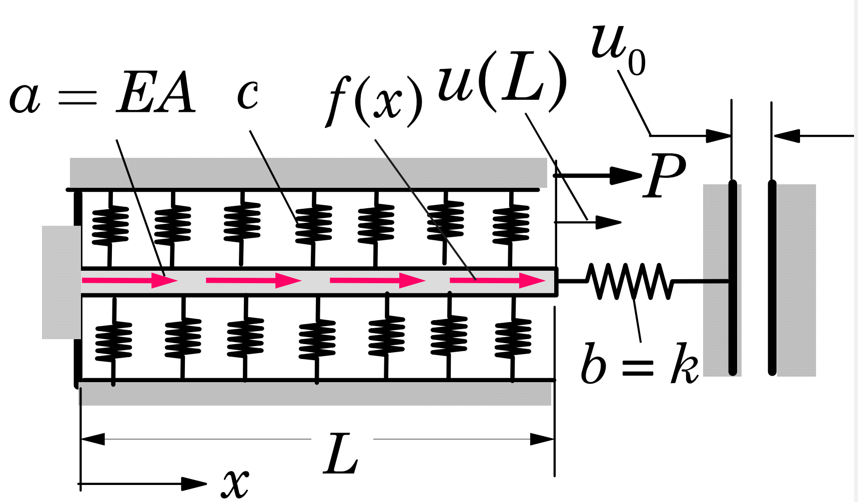
Data (i.e., information you need to solve the problem)

$$L,\ a(x),\ c(x),\ f(x),\ b,\ \hat{u},\ u_0,\ P$$

ENGINEERING EXAMPLES OF THE MODEL PROBLEM IN 1-D

Elastic deformation of a bar

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$



$u(x)$ = axial displacement

$f(x)$ = body force

$E(x)$ = modulus of elasticity

$A(x)$ = area of cross section

k = spring elastic (linear) stiffness

c = surface shear resistance (linear)

P = axial point load at $x = L$

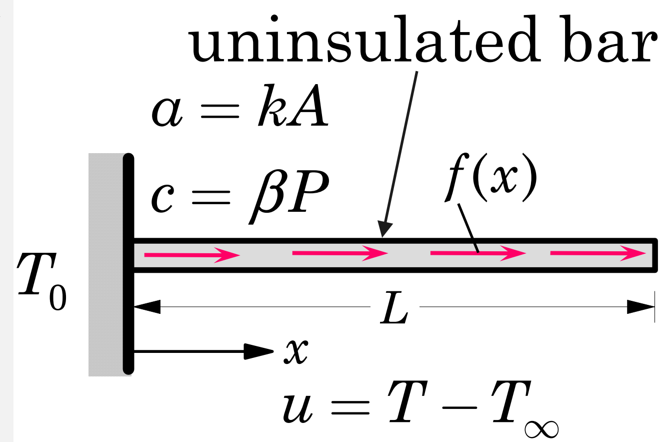
$$u(0)=0 \qquad \left[EA\frac{du}{dx}+k(u-u_0)\right]_{x=L}=P$$

ENGINEERING EXAMPLES OF THE MODEL PROBLEM IN 1-D

1D Heat flow through a fin

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$

$u(x)=T(x)-T_\infty$, absolute temperature
$f(x)=$ internal heat generation
$k(x)=$ thermal conductivity
$A(x)=$ area of cross section
$\beta=$ heat transfer coefficient
$P=$ perimeter
$Q=$ heat input at $x=L$
$T_\infty=$ ambient temperature



$$u(0)=T_0-T_\infty,\qquad \left[kA\frac{du}{dx}+\beta u\right]_{x=L}=\hat{Q}$$

ENGINEERING EXAMPLES OF THE MODEL PROBLEM IN 1-D

Flow of a viscous fluid through a channel

$$-\frac{d}{dy}\left(a\frac{du}{dy}\right)-f=0 \text{ in } \Omega=(0,b)$$

$u(y)=v_x$, axial velocity

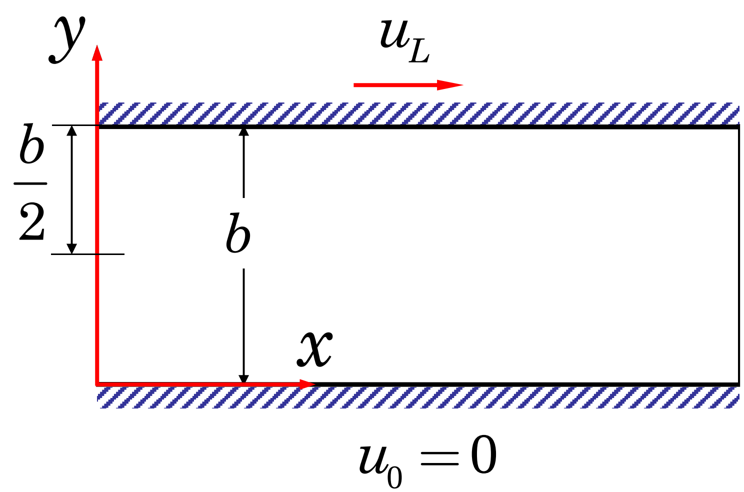
$f=-\dfrac{dp}{dx}$, pressure gradient

$a=\mu=$ viscosity of the fluid

$Q=\tau_L$ = shear stress at $x=L$

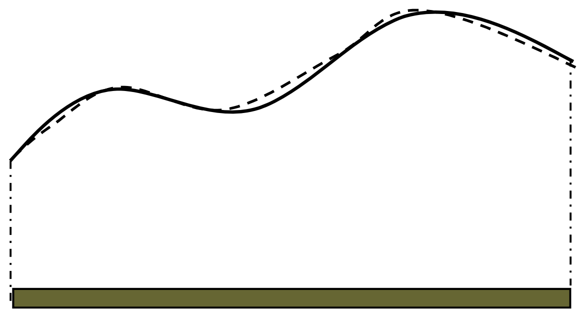
$u_L=0$, Poiseuille flow

$u_L\neq 0$, Couette flow

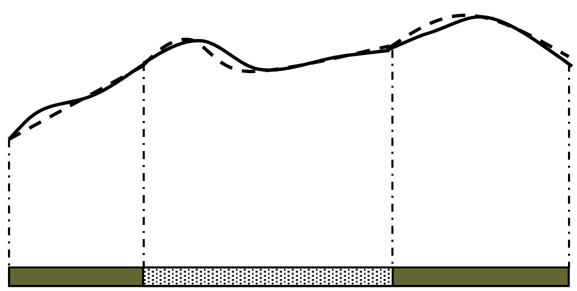


$$u(0)=u_0,\quad u(b)=u_L$$

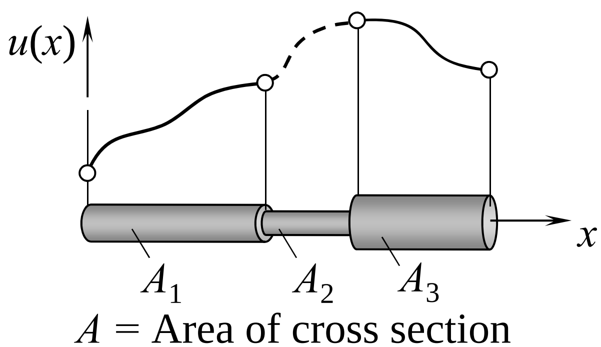
Finite Element Approximation
(Need for Seeking Solution on Sub-intervals)

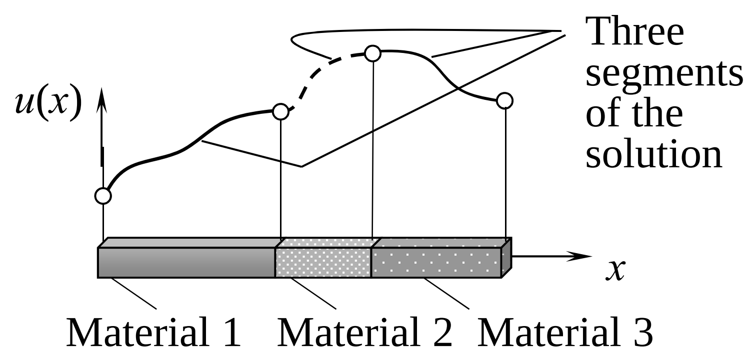
1. 

Approximation of the actual solution over the entire domain requires higher-order (or degree) polynomials.

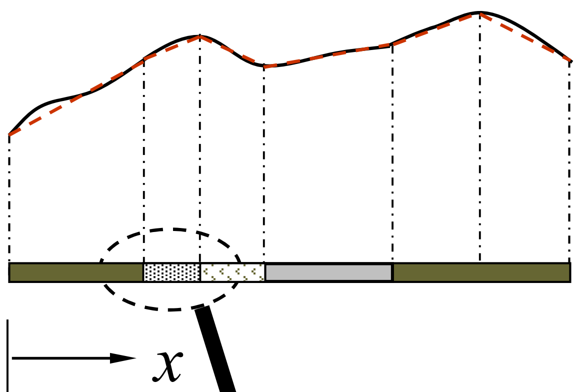
2. 

Actual solution may be defined by piecewise continuous functions because of discontinuity of the data.



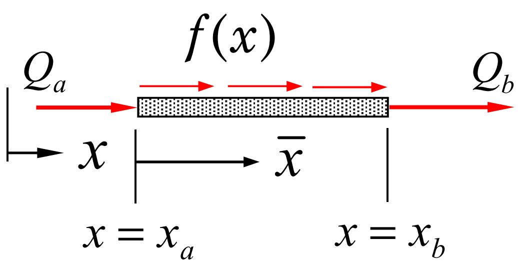


FINITE ELEMENT DISCRETIZATION



Approximation over subintervals (or elements) allows lower-order approximation of the actual solution

A "free-body-diagram" of typical element (geometry and 'forces')



Q_a, Q_b end forces or heats (scalar)

$h = x_b - x_a$ = element length

Approximate Solution and Residual of the Approximation

Approximate solution: $u(x) \approx u_h(x)$

$$\left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)+c(x)u_h-f(x)\right]=R(x)\neq 0$$

We seek to make $R(x)$ zero in a weighted-residual sense:

$$0=\int_{x_a}^{x_b} w_i\, R(x)\,dx,$$

w_i is the weight function from a set of weight functions $\{w_i\}$

$$=\int_{x_a}^{x_b} w_i\left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)+c(x)u_h-f(x)\right]dx$$

Trading of Differentiation between the weight function and the variable

Product rule of differentiation (an identity) and integration-by-parts

$$\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right)=\frac{dw_i}{dx}a(x)\frac{du_h}{dx}+w_i\,\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)$$

$$-w_i\,\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)=-\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right)+\frac{dw_i}{dx}a(x)\frac{du_h}{dx}$$

$$\int_{x_a}^{x_b}-w_i\,\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)dx=\int_{x_a}^{x_b}\left[-\frac{d}{dx}\left(w_i\, a(x)\frac{du_h}{dx}\right)+\frac{dw_i}{dx}a(x)\frac{du_h}{dx}\right]dx$$

$$=\int_{x_a}^{x_b}a(x)\frac{dw_i}{dx}\frac{du_h}{dx}\,dx-\left(w_i\, a(x)\frac{du_h}{dx}\right)_{x_a}^{x_b}$$

Identification of the Primary and Secondary Variables

Examine the boundary term(s) obtained during integration-by-parts:

$$\left[w_i \cdot a \frac{du_h}{dx} \right]_{x_a}^{x_b}$$

Secondary variable

The expression always contains the weight function w and a coefficient that depends on the dependent unknown. In this case the coefficient is $a(du_h/dx)$. We will term the coefficient a <u>secondary variable</u> (a name we choose to give).

The weight function w_i in the boundary term when replaced with the dependent variable u_h of the problem is termed the <u>primary variable</u>.

$u_h \longrightarrow$ Primary variable

$$\left[w_i \cdot a \frac{du_h}{dx} \right]_{x_a}^{x_b}$$

WEAK FORM DEVELOPMENT OVER AN ELEMENT

$$0=\int_{x_a}^{x_b} w_i\left[-\frac{d}{dx}\left(a(x)\frac{du_h}{dx}\right)+c(x)u_h-f(x)\right]dx \qquad w-\text{weight function}$$

$$=\int_{x_a}^{x_b}\left[a\frac{dw_i}{dx}\frac{du_h}{dx}+cw_iu_h-w_if\right]dx-\left[w_i\cdot a\frac{du_h}{dx}\right]_{x_a}^{x_b} \qquad \text{secondary variable}$$

$$=\int_{x_a}^{x_b}\left[a\frac{dw_i}{dx}\frac{du_h}{dx}+cw_iu_h-w_if\right]dx-w_i(x_a)\cdot\left(-a\frac{du_h}{dx}\right)_{x_a}-w_i(x_b)\cdot\left(a\frac{du_h}{dx}\right)_{x_b}$$

$$=\int_{x_a}^{x_b}\left[a\frac{dw_i}{dx}\frac{du_h}{dx}+cw_iu_h-w_if\right]dx-w_i(x_a)Q_{\mathbf{a}}-w_i(x_b)\cdot Q_b$$

Final Weak Form

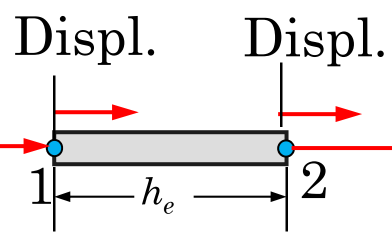
$$\boxed{0=\int_{x_a}^{x_b}\left[a\frac{dw_i}{dx}\frac{du_h}{dx}+cw_iu_h-w_if\right]dx-w_i(x_a)Q_{\mathbf{a}}-w_i(x_b)\cdot Q_b}$$

Interpretation of the secondary variables

Axial deformation of a bar

Force

$$\left(-a\frac{du_h}{dx}\right)_{x_a} \equiv Q_a$$



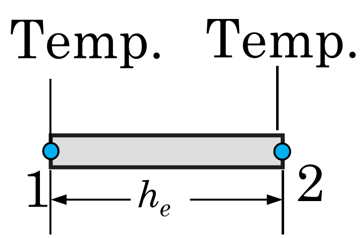
Force

$$\left(a\frac{du_h}{dx}\right)_{x_b} \equiv Q_b$$

Axial heat flow

Heat input

$$\left(-a\frac{du_h}{dx}\right)_{x_a} \equiv Q_a$$



Heat input

$$\left(a\frac{du_h}{dx}\right)_{x_b} \equiv Q_b$$

Primary and Secondary Variables
(Some Remarks)

Primary variables and secondary variables always appear in pairs. They are like `cause' and `effect' (i.e., one is the result of the other). For example, when u_h is the temperature, $a(du_h/dx)$ is heat (and heat causes temperature). When u_h is the displacement, $a(du_h/dx)$ is the force. This **duality** exists in every engineering problem.

Essential and Natural Boundary Conditions

Essential Boundary Conditions: Specifying a primary variable at a boundary point of the domain is called an essential (or Dirichlet) boundary condition.

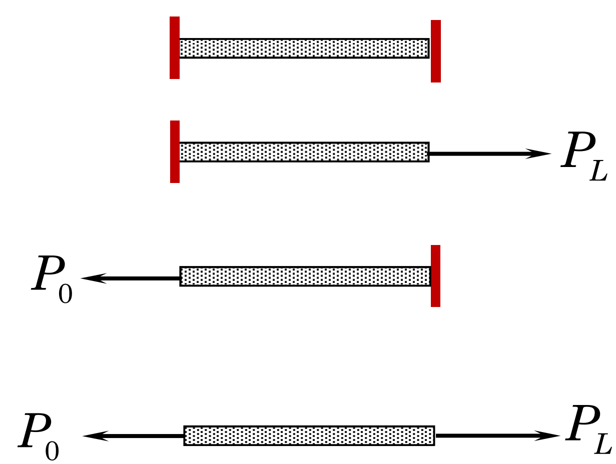
Natural Boundary Conditions: Specifying a secondary variable at a boundary point of the domain is called a natural (or Neumann) boundary condition.

Essential and Natural Boundary Conditions

Recall that the primary and secondary variables appear as pairs. One may specify only one element (variable) of each pair at a boundary point. Thus, when $[u, a(du/dx)]$ is the pair, only u or $a(du/dx)$, and <u>never,</u> both may be specified at a boundary point.

Thus, for a problem with two boundary points, there are four combinations of boundary conditions:

1. $u(0) = u_0, \qquad u(L) = u_L$
2. $u(0) = u_0, \qquad \left(a\dfrac{du}{dx}\right)_{x=L} = P_L$
3. $\left(a\dfrac{du}{dx}\right)_{x=0} = P_0, \qquad u(L) = u_L$
4. $\left(a\dfrac{du}{dx}\right)_{x=0} = P_0, \ \left(a\dfrac{du}{dx}\right)_{x=L} = P_L \ \ (P_0 = P_L)$



Linear, Bilinear Forms, and the Variational Problem

Weak Form

$$0 = \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h - w_i f \right] dx - w_i(x_a) Q_a - w_i(x_b) \cdot Q_b$$

$$= \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h \right] dx - \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b) \cdot Q_b \right]$$

$$= B(w_i, u_h) - l(w_i)$$

Bilinear Form and Linear Form

$$B(w_i, u_h) = \int_{x_a}^{x_b} \left[a \frac{dw_i}{dx}\frac{du_h}{dx} + cw_i u_h \right] dx, \quad l(w_i) = \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b) \cdot Q_b \right]$$

Variational Problem: Find u such that

$$B(w_i, u_h) = l(w_i) \quad \text{holds for all} \quad w_i$$

Equivalence Between Minimum of a the Total Potential Energy and Weak Form

Total potential energy (of uniaxial members):

$$\Pi = U + W_E$$

$$= \frac{1}{2}\int_{x_a}^{x_b}\left[EA\left(\frac{du}{dx}\right)^2 + c_f u^2\right]dx - \left(\int_{x_a}^{x_b} uf\ dx + \sum_{i=1}^{n} u_i^e Q_i^e\right)$$

$$\delta\Pi = \int_{x_a}^{x_b}\left[EA\left(\frac{d\delta u}{dx}\right)\left(\frac{du}{dx}\right) + c_f \delta u\, u\right]dx - \left(\int_{x_a}^{x_b} \delta u\, f\ dx + \sum_{i=1}^{n} \delta u_i^e\, Q_i^e\right)$$

Now let $\delta u = w_i$. Then $\delta\Pi = 0$ gives the weak form:

$$0 = \int_{x_a}^{x_b}\left[EA\left(\frac{dw_i}{dx}\right)\left(\frac{du}{dx}\right) + c_f w_i\, u\right]dx - \left(\int_{x_a}^{x_b} w_i\, f\ dx + \sum_{j=1}^{n} w_i(x_j) Q_j^e\right)$$

Equivalence Between Minimum of a Quadratic Functional and Weak Form

Replace w with δu

$$0 = \int_{x_a}^{x_b} \left[a \frac{d\delta u}{dx}\frac{du}{dx} + c\delta u\, u - \delta u\, f \right] dx - \delta u(x_a) Q_a - \delta u(x_b) Q_b$$

$$= \int_{x_a}^{x_b} \left[a \frac{d\delta u}{dx}\frac{du}{dx} + c\delta u\, u \right] dx - \left[\int_{x_b}^{x_a} \delta u\, f\, dx + \delta u(x_a) Q_a + \delta u(x_b) Q_b \right]$$

$$= \frac{1}{2}\delta \int_{x_a}^{x_b} \left[a\left(\frac{du}{dx}\right)^2 + c\, u^2 \right] dx - \delta \left[\int_{x_b}^{x_a} u\, f\, dx + u(x_a) Q_a + u(x_b) Q_b \right]$$

$$= \delta\left[\tfrac{1}{2} B(u,u) - l(u)\right] = \delta I(u)$$

or [when $B(w,u)$ is bilinear and symmetric]

$$\boxed{I(u) = \tfrac{1}{2} B(u,u) - l(u)}$$

FINITE ELEMENT MODEL

(is a set of algebraic relations between the primary and the secondary variables at the nodes)

Finite element approximation $u(x) \approx u_h^e(x) = \sum_{j=1} u_j^e \psi_j^e(x)$

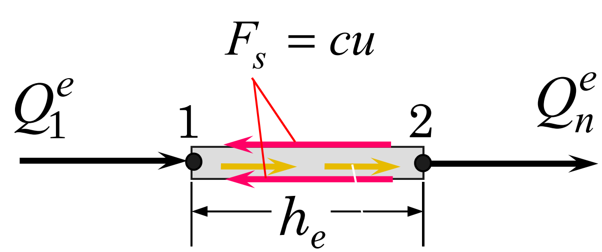
$w_i(x) = \psi_i^e(x)$

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw_i}{dx} \frac{du_h}{dx} + c w_i u_h \right) dx - \left[\int_{x_b}^{x_a} w_i f \, dx + w_i(x_a) Q_a + w_i(x_b) \cdot Q_b \right]$$

$$= \sum_{j=1}^{n} u_j \int_{x_a}^{x_b} \left[a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c \psi_i \psi_j \right] dx - \left[\int_{x_b}^{x_a} \psi_i f \, dx + \psi_i(x_a) Q_1^e + \psi_i(x_b) Q_n^e \right]$$

$$\sum_{j=1}^{n} K_{ij}^e u_j^e = F_i^e \quad \Rightarrow \quad [K^e]\{u^e\} = \{F^e\}$$

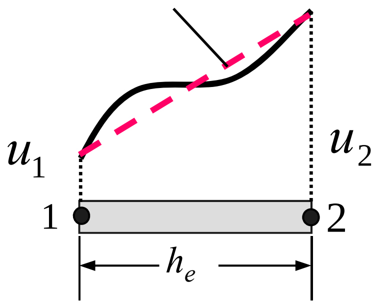
$$K_{ij}^e = B(\psi_i, \psi_j) = \int_{x_a}^{x_b} \left(a_e \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx,$$



$$F_i^e = l(\psi_i) = \int_{x_a}^{x_b} f_e \psi_i \, dx + \psi_i(x_1^e) Q_1^e + \psi_i(x_2^e) Q_2^e + \cdots + \psi_i(x_n^e) Q_n^e$$

Approximation (interpolation or shape) Functions for *Linear* Element

$$u(x) \approx u_h(x) = c_1 + c_2 x$$



$$u_h(x) = c_1 + c_2 x$$

Rewrite c_1 and c_2 in terms of u_1 and u_2

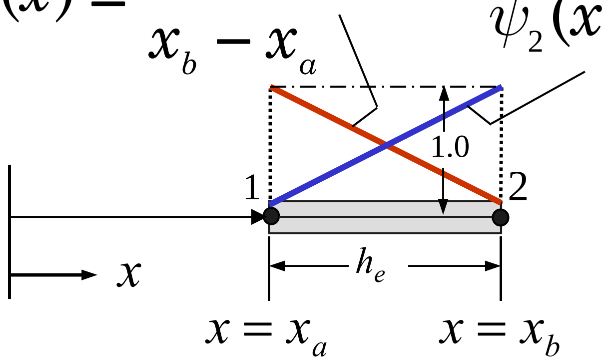
$$u_h(x_b) \equiv u_b = u_2 \quad u_h(x_a) \equiv u_a = u_1$$

$$u_h(x_a) = u_1 = c_1 + c_2 x_a; \quad u_h(x_b) = u_2 = c_1 + c_2 x_b$$

$$\rightarrow c_1 = \frac{u_1 x_b - u_2 x_a}{x_b - x_a}, \quad c_2 = \frac{u_2 - u_1}{x_b - x_a}$$

$$\psi_1(x) \equiv \frac{x_b - x}{x_b - x_a}$$

$$\psi_2(x) \equiv \frac{x - x_a}{x_b - x_a}$$



$$u(x) \approx u_h(x) = c_1 + c_2 x$$

$$= \frac{u_1 x_b - u_2 x_a}{x_b - x_a} + \frac{u_2 - u_1}{x_b - x_a} x$$

$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

Interpolation Properties of the Approximation Functions

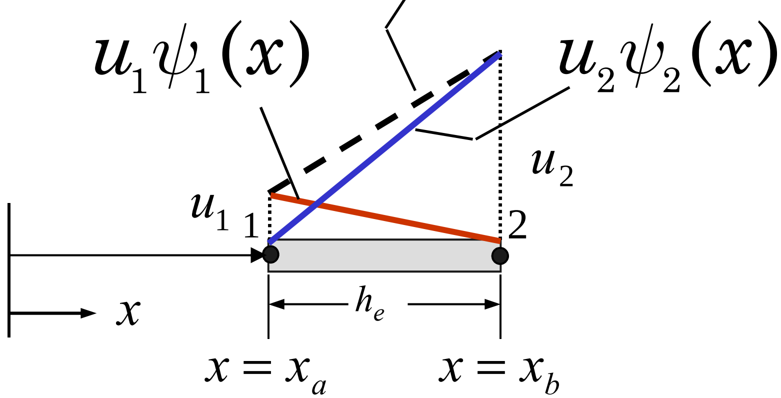
$$\psi_i(x_j) \equiv \delta_{ij} = \begin{cases} 0, & if \;\; i \neq j \\ 1, & if \;\; i = j \end{cases} \qquad x_1 \equiv x_a, \;\; x_2 \equiv x_b$$

Interpolation property

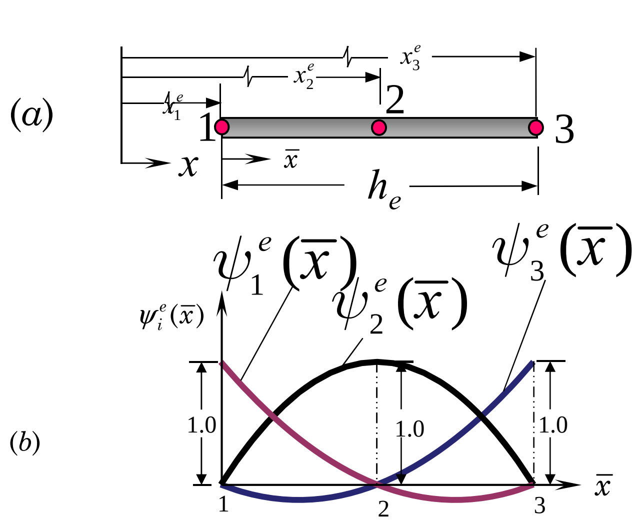
$$u_h(x) = \psi_1(x)u_1 + \psi_2(x)u_2$$

Partition of unity

$$\sum_{j=1}^{n} \psi_j(x) = 1$$



Derivation of Approximation Functions
(*Quadratic* Element)



$$u(x) \approx u_h(x) = c_1 + c_2 x + c_3 x^2$$

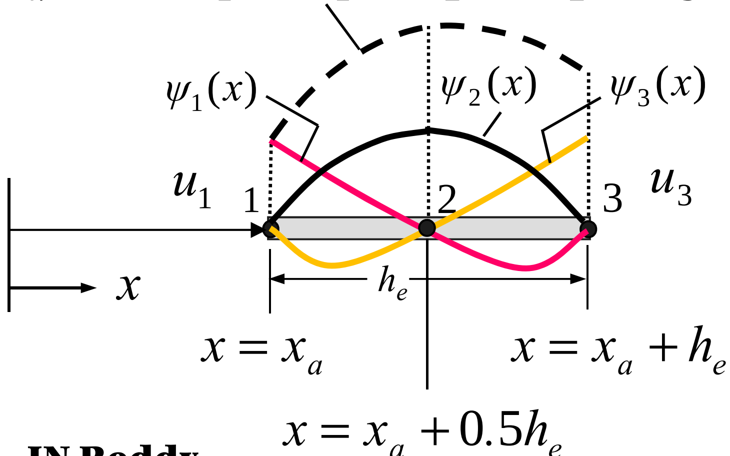
Derivation using the interpolation property

$$\psi_1(\bar{x}) = \alpha_1 (h - \bar{x})(0.5h - \bar{x}),\ \psi_1(0) = 1 \rightarrow \alpha_1 = \frac{2}{h^2}$$

$$\psi_2(\bar{x}) = \alpha_2 (h - \bar{x})(\bar{x} - 0),\ \psi_2(0.5h) = 1 \rightarrow \alpha_2 = \frac{4}{h^2}$$

$$\psi_3(\bar{x}) = \alpha_3 (\bar{x} - 0)(0.5h - \bar{x}),\ \psi_3(h) = 1 \rightarrow \alpha_3 = -\frac{2}{h^2}$$

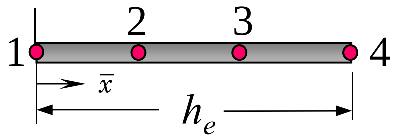
$$u_h(x) = \psi_1(x) u_1 + \psi_2(x) u_2 + \psi_3(x) u_3$$



$$\psi_1(\bar{x}) = \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{2\bar{x}}{h}\right),\ \psi_2(\bar{x}) = 4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right)$$

$$\psi_3(\bar{x}) = -\frac{\bar{x}}{h}\left(1 - \frac{2\bar{x}}{h}\right)$$

Derivation of Approximation Functions
(*Cubic* Element)



$$u(x) \approx u_h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

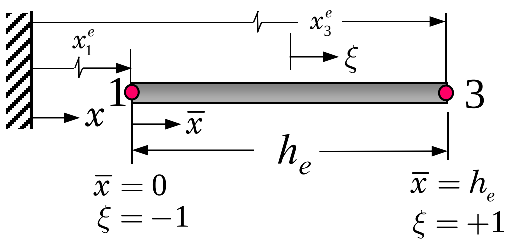
Derivation using the interpolation property

$$\psi_1(\bar{x}) = \alpha_1 \left(\bar{x} - \frac{h}{3}\right)\left(\bar{x} - \frac{2h}{3}\right)(\bar{x} - h)$$

$$\psi_1(0) = 1 \rightarrow \alpha_1 = -\frac{9}{2h^3}$$

$$\psi_1(\bar{x}) = \left(1 - \frac{3\bar{x}}{h}\right)\left(1 - \frac{3\bar{x}}{2h}\right)\left(1 - \frac{\bar{x}}{h}\right)$$

Approximation Functions in terms of the local coordinates



Relations between various coordinates

$$x = \bar{x} + x_1^e$$

$$x = x_1^e \tfrac{1}{2}(1-\xi) + x_2^e \tfrac{1}{2}(1+\xi)$$

$$= \frac{x_1^e + x_2^e}{2} + \frac{h_e}{2}\xi$$

$$\left.\begin{aligned} \psi_1(x) &= \frac{1}{h_e}(x_2^e - x), \\ \psi_2(\bar{x}) &= \frac{1}{h_e}(x - x_1^e) \end{aligned}\right\} \quad x_1^e < x < x_2^e$$

$$\psi_1(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \ \psi_2(\bar{x}) = \frac{\bar{x}}{h_e}, \qquad 0 < \bar{x} < h_e$$

$$\psi_1(\xi) = \tfrac{1}{2}(1-\xi), \ \psi_2(\bar{x}) = \tfrac{1}{2}(1+\xi), \ -1 < \xi < 1$$

NUMERICAL EVALUATION OF COEFFICIENTS in various coordinates

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} \left(a(x) \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c(x)\psi_i(x)\psi_j(x) \right) dx$$

$$= \int_0^{h_e} \left(a(\bar{x}) \frac{d\psi_i}{d\bar{x}} \frac{d\psi_j}{d\bar{x}} + c(\bar{x})\psi_i(\bar{x})\psi_j(\bar{x}) \right) d\bar{x}$$

$$F_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx + \ Q_i = \int_0^{h_e} f(\bar{x})\psi_i(\bar{x})\, d\bar{x} + \ Q_i$$

$$F_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx + \ Q_i = \int_{-1}^{+1} f(\xi)\psi_i(\xi)\, J d\xi + \ Q_i$$

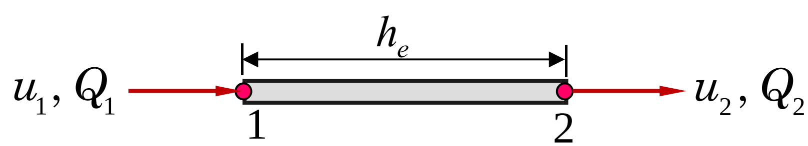
$$J = \frac{dx}{d\xi} = \frac{h_e}{2}$$

NUMERICAL EVALUATION OF COEFFICIENTS
for element-wise constant data

$$\text{For constant } data: \quad a = a_e,\ c = c_e,\ f = f_e$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c\psi_i \psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f \psi_i \, dx + Q_i$$

Linear element:



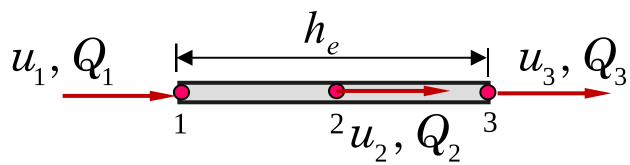
$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

NUMERICAL EVALUATION OF COEFFICIENTS
for element-wise constant data

$$\text{For constant}\, data: \quad a = a_e,\, c = c_e,\, f = f_e$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c\psi_i\psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f\psi_i \; dx + \; Q_i$$

Quadratic element:

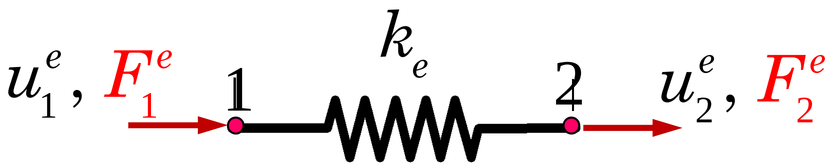


known

$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad \left\{F^e\right\} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ \cancel{Q_2} \\ Q_3 \end{Bmatrix}$$

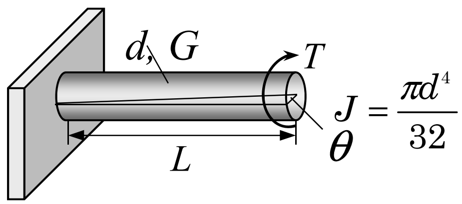
Other (Discrete) Elements

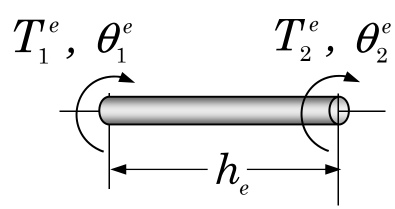
Spring element:



$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix}$$

Torsion element:



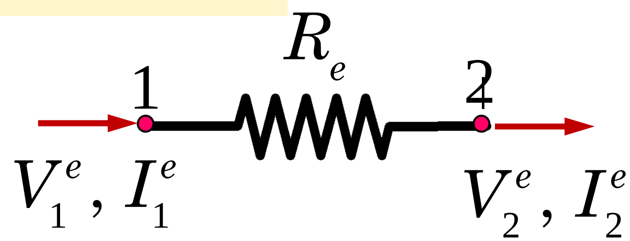


$$[K^e] = \frac{G_e J_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix}$$

Other (Discrete) Elements

(continued)

Electrical element: $k_e = \dfrac{1}{R_e}$

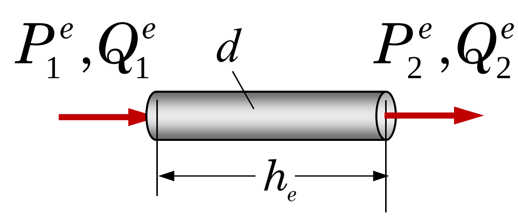


$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} I_1^e \\ I_2^e \end{Bmatrix}$$

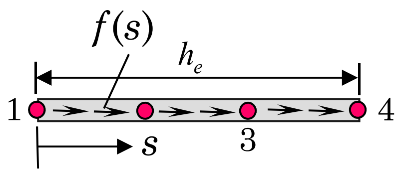
$R_e =$ Electrical resistance

Pipe flow element: $k_e = \dfrac{\pi d^4}{128 \mu h_e}$

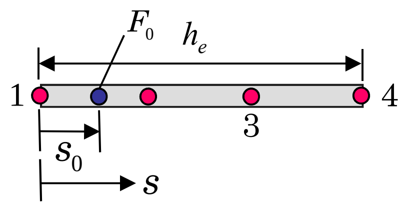
$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$



Representation of Point Sources at points other than nodes

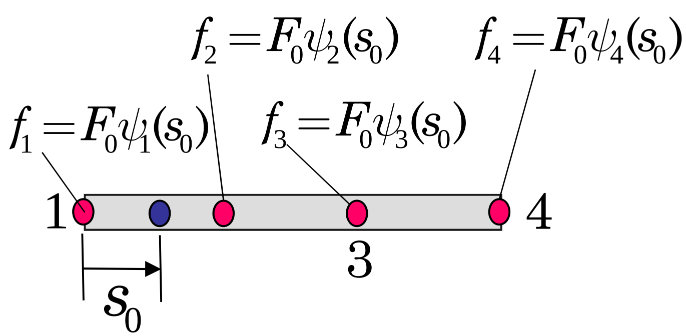


$$f_i^e = \int_0^h f(s)\,\psi_i(s)\,ds$$

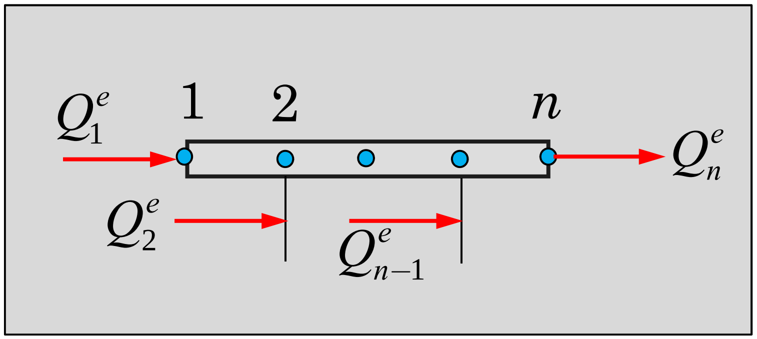


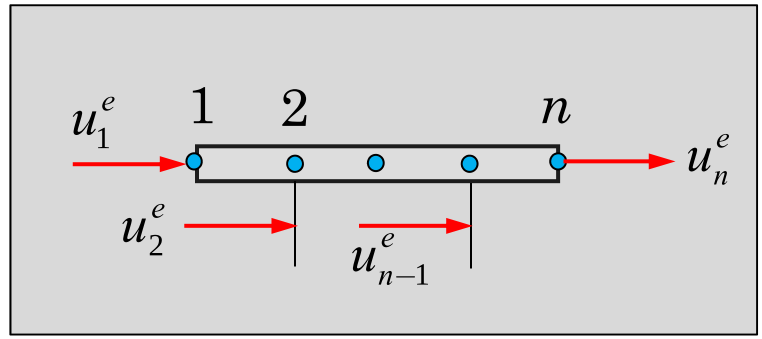
$$f(s) = F_0\,\delta(s - s_0), \qquad f_i^e = \int_0^h f(s)\,\psi_i(s)\,ds$$

$$f_i^e = \int_0^h F_0\,\delta(s - s_0)\,\psi_i(s)\,ds = F_0\psi_i(s_0)$$



Equivalence Between Minimum of a the Total Potential Energy and Weak Form





Strain energy:

$$U = \frac{1}{2}\sum_{i,j=1}^{3} \int_{V^e} \sigma_{ij}\, \varepsilon_{ij}\; dv = \frac{1}{2}\int_{V^e} \sigma_{xx}\, \varepsilon_{xx}\; dv = \frac{1}{2}\int_{V^e} E\,(\varepsilon_{xx})^2\; dv$$

$$= \frac{1}{2}\int_{V^e} E\left(\frac{du}{dx}\right)^2 dv = \frac{1}{2}\int_{x_a}^{x_b} EA\left(\frac{du}{dx}\right)^2\; dx$$

Work done by external forces:

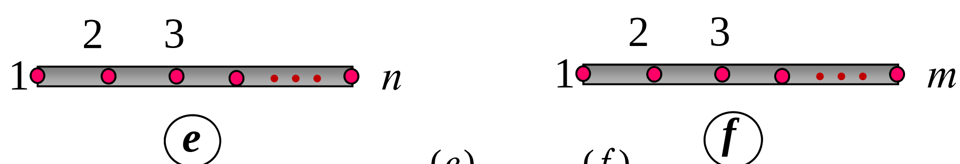
$$W_E = -\left(\int_{x_a}^{x_b} uf\; dx + \sum_{i=1}^{n} u_i^e Q_i^e\right)$$

ASSEMBLY OF ELEMENTS

Assembly of elements is based on two requirements:

- **Continuity** of the primary variables across the element boundaries.
- **Balance** of the secondary variables between the element boundaries.

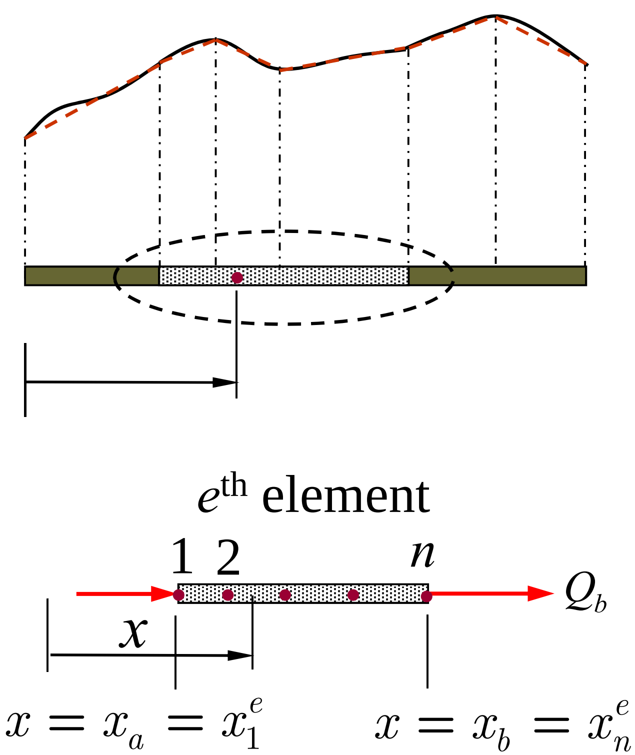
(1)



$$u_n^{(e)} = u_1^{(f)}$$

(2) $Q_n^{(e)} + Q_1^{(f)} = 0$ or equal to externally applied source

POST-COMPUTATION OF VARIABLES



$$
u_h^e = \begin{cases}
\sum_{j=1}^{n} u_j^{(1)} \psi_j^{(1)}(x), & x_1^{(1)} \leq x \leq x_n^{(1)} \\
\sum_{j=1}^{n} u_j^{(2)} \psi_j^{(2)}(x), & x_1^{(2)} \leq x \leq x_n^{(2)} \\
\cdots\cdots & \cdots\cdots \quad \cdots\cdots \\
\sum_{j=1}^{n} u_j^{(N)} \psi_j^{(N)}(x), & x_1^{(N)} \leq x \leq x_n^{(N)}
\end{cases}
$$

$$
\frac{du_h^e}{dx} = \begin{cases}
\sum_{j=1}^{n} u_j^{(1)} \dfrac{d\psi_j^{(1)}}{dx}, & x_1^{(1)} \leq x \leq x_n^{(1)} \\
\sum_{j=1}^{n} u_j^{(2)} \dfrac{d\psi_j^{(2)}}{dx}, & x_1^{(2)} \leq x \leq x_n^{(2)} \\
\cdots\cdots & \cdots\cdots \quad \cdots\cdots \\
\sum_{j=1}^{n} u_j^{(N)} \dfrac{d\psi_j^{(1)}}{dx}, & x_1^{(N)} \leq x \leq x_n^{(N)}
\end{cases}
$$

A DIFFERENTIAL EQUATION

1 **Problem**: Wish to determine the numerical solution of the differential equation

$$-\frac{d^2u}{dx^2} - u = -x^2 \ \text{ in } \ 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

FE Solution: We have the following correspondence compared to the model equation:

$$a = 1, \ c = -1, \ f = -x^2, \ f_i^e = -\int_{x_a}^{x_b} x^2 \psi_i \ dx$$

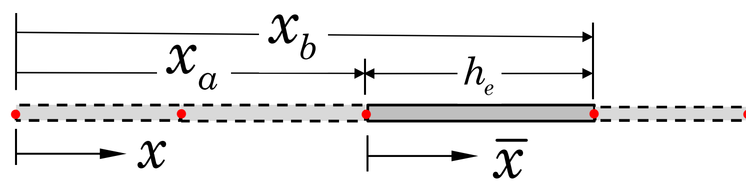
(1) We wish to use a mesh of linear elements to solve the problem. The equations of a typical element are

$$\left(\frac{1}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} f_1^e \\ f_2^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

Details of the computation of the nodal source vector due to the distributed source

$$f_i^e = \int_{x_a}^{x_b} f(x)\psi_i(x)\, dx = -\int_{x_a}^{x_b} x^2 \psi_i\, dx = -\int_0^{h_e} (\bar{x} + x_a)^2 \psi_i(\bar{x})\, d\bar{x}$$

$$\psi_1(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2(\bar{x}) = \frac{\bar{x}}{h_e}$$



$$x = \bar{x} + x_a$$

$$f_i^e = -\int_0^{h_e} (\bar{x} + x_a)^2 \psi_i(\bar{x})\, d\bar{x} = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\, x_a + x_a^2\right) \psi_i(\bar{x})\, d\bar{x}$$

$$f_1^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\, x_a + x_a^2\right) \left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x}$$

$$f_2^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\, x_a + x_a^2\right) \left(\frac{\bar{x}}{h_e}\right) d\bar{x}$$

Details of the computation of the nodal source vector due to the distributed source

$$f_1^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x}$$

$$= -\left[\frac{\bar{x}^3}{3} + \bar{x}^2 x_a + \bar{x}\,x_a^2\right]_0^{h_e} + \frac{1}{h_e}\left[\frac{\bar{x}^4}{4} + \frac{2\bar{x}^3}{3}x_a + \frac{\bar{x}^2}{2}x_a^2\right]_0^{h_e}$$

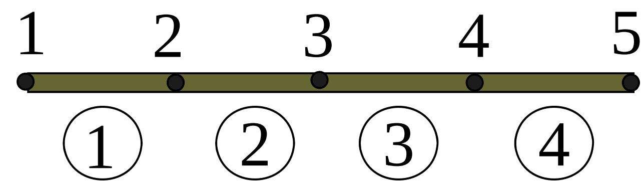
$$= -\left[\frac{h_e^3}{3} + h_e^2 x_a + h_e\,x_a^2\right] + \frac{1}{h_e}\left[\frac{h_e^4}{4} + \frac{2h_e^3}{3}x_a + \frac{h_e^2}{2}x_a^2\right]$$

$$f_2^e = -\int_0^{h_e} \left(\bar{x}^2 + 2\bar{x}\,x_a + x_a^2\right)\left(\frac{\bar{x}}{h_e}\right) d\bar{x} = -\frac{1}{h_e}\left[\frac{h_e^4}{4} + \frac{2h_e^3}{3}x_a + \frac{h_e^2}{2}x_a^2\right]$$

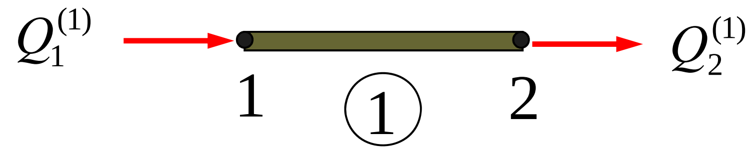
Note that $x_a = 0$ for Element 1, $x_a = h_1$ for Element 2, $x_a = h_1 + h_2$ for Element 3, and $x_a = h_1 + h_2 + h_3$ for Element 4

A DIFFERENTIAL EQUATION (cont.)

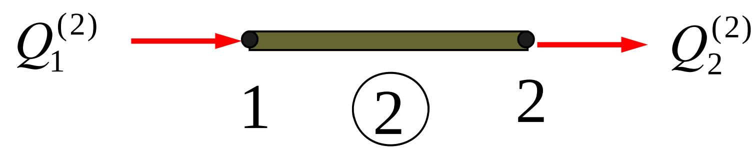
(2) We consider a mesh of 4 linear elements (h =0.25).
The element equations are



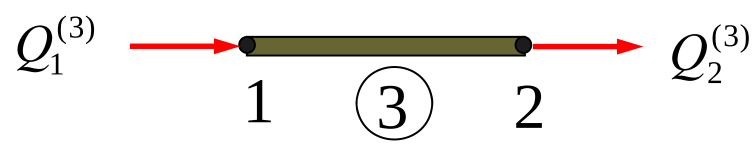
$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = -\begin{Bmatrix} 0.00130 \\ 0.00391 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$



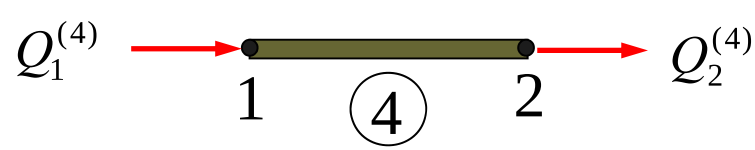
$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = -\begin{Bmatrix} 0.01432 \\ 0.02232 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$



$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = -\begin{Bmatrix} 0.04297 \\ 0.05599 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$



$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix} = -\begin{Bmatrix} 0.08724 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

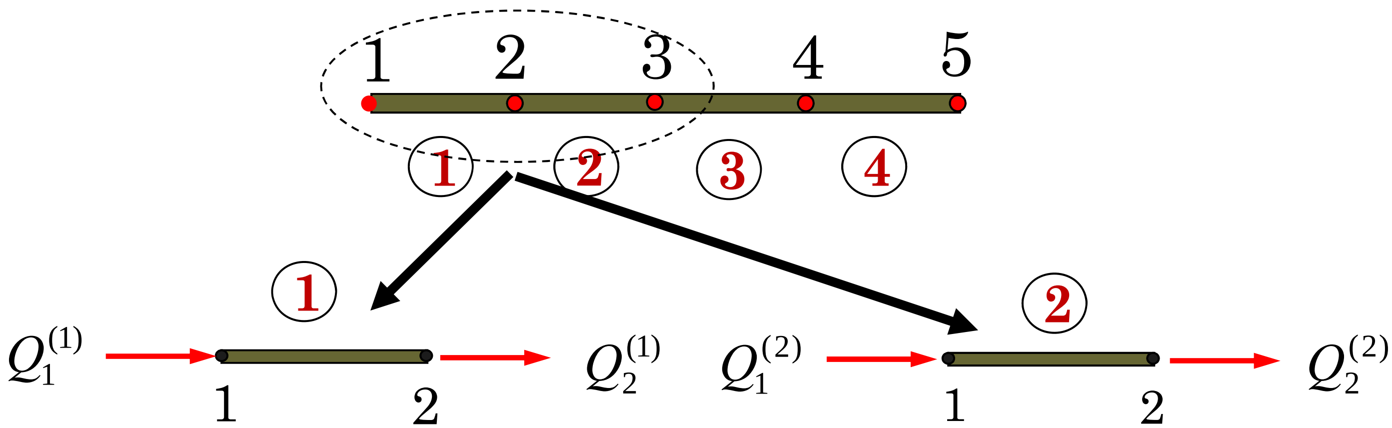


A DIFFERENTIAL EQUATION (cont.)

(3) The boundary conditions on the primary variables are

$$U_1 = 0, \quad U_5 = 0$$

The equilibrium conditions are



$$Q_2^{(1)} + Q_1^{(2)} = 0, \; Q_2^{(2)} + Q_1^{(3)} = 0, \; Q_2^{(3)} + Q_1^{(4)} = 0$$

A DIFFERENTIAL EQUATION (cont.)

(4) The assembled equations are

$$\frac{1}{24}\begin{bmatrix} 94 & -97 & 0 & 0 & 0 \\ -97 & 188 & -97 & 0 & 0 \\ 0 & -97 & 188 & -97 & 0 \\ 0 & 0 & -97 & 188 & -97 \\ 0 & 0 & 0 & -97 & 94 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = -\begin{Bmatrix} 0.001302 \\ 0.018229 \\ 0.065104 \\ 0.143230 \\ 0.105470 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

0 0 0 0 0

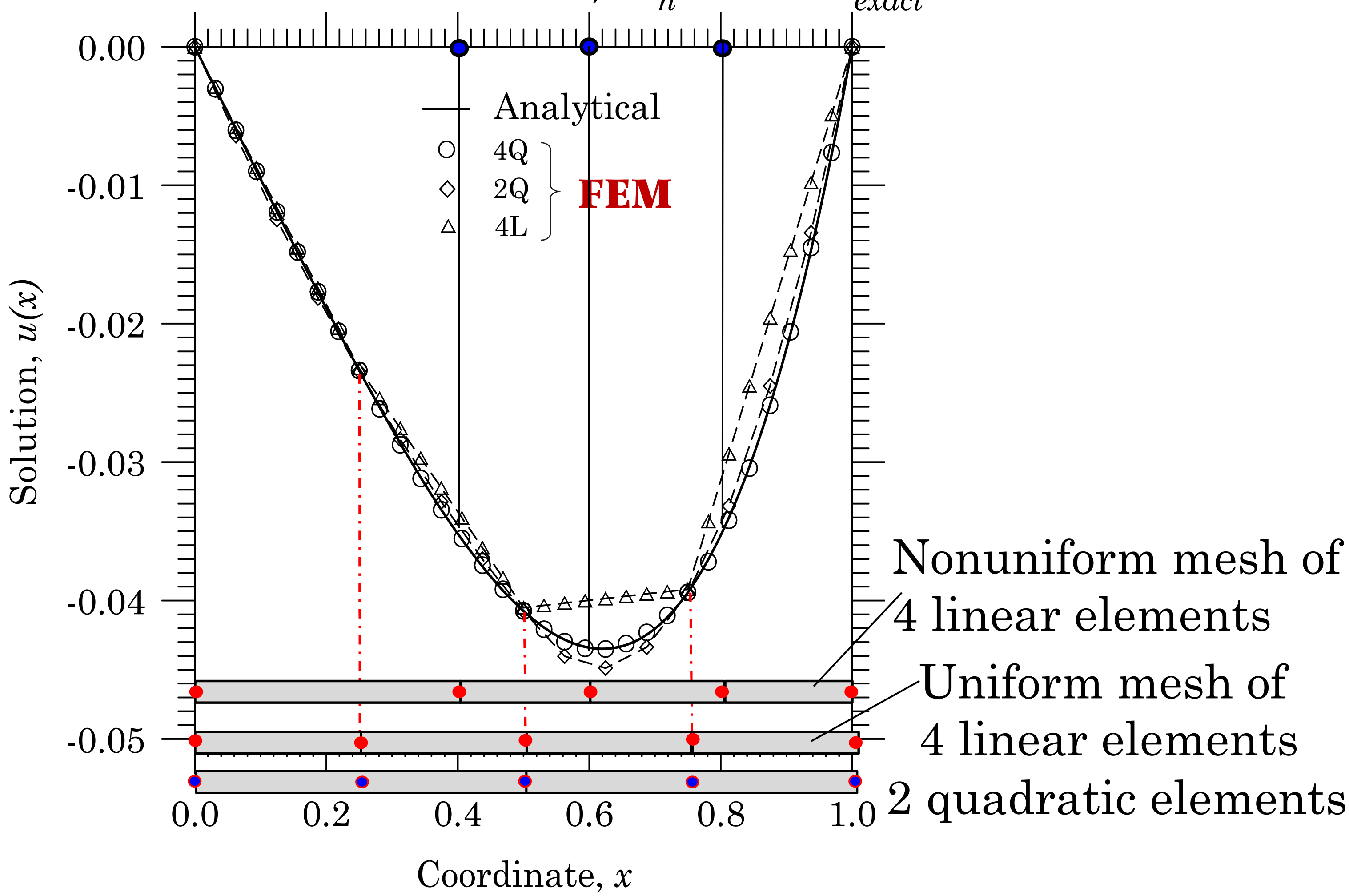
(5) The condensed equations for the unknown U's and Q's are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^1 \\ Q_2^4 \end{Bmatrix} = \begin{bmatrix} -4.0417 & 0 & 0 \\ 0 & 0 & -4.0417 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} 0.00130 \\ 0.10547 \end{Bmatrix} = \begin{Bmatrix} 0.09520 \\ 0.26386 \end{Bmatrix}$$

A DIFFERENTIAL EQUATION (cont.)

Plot of the solution, u_h and u_{exact}



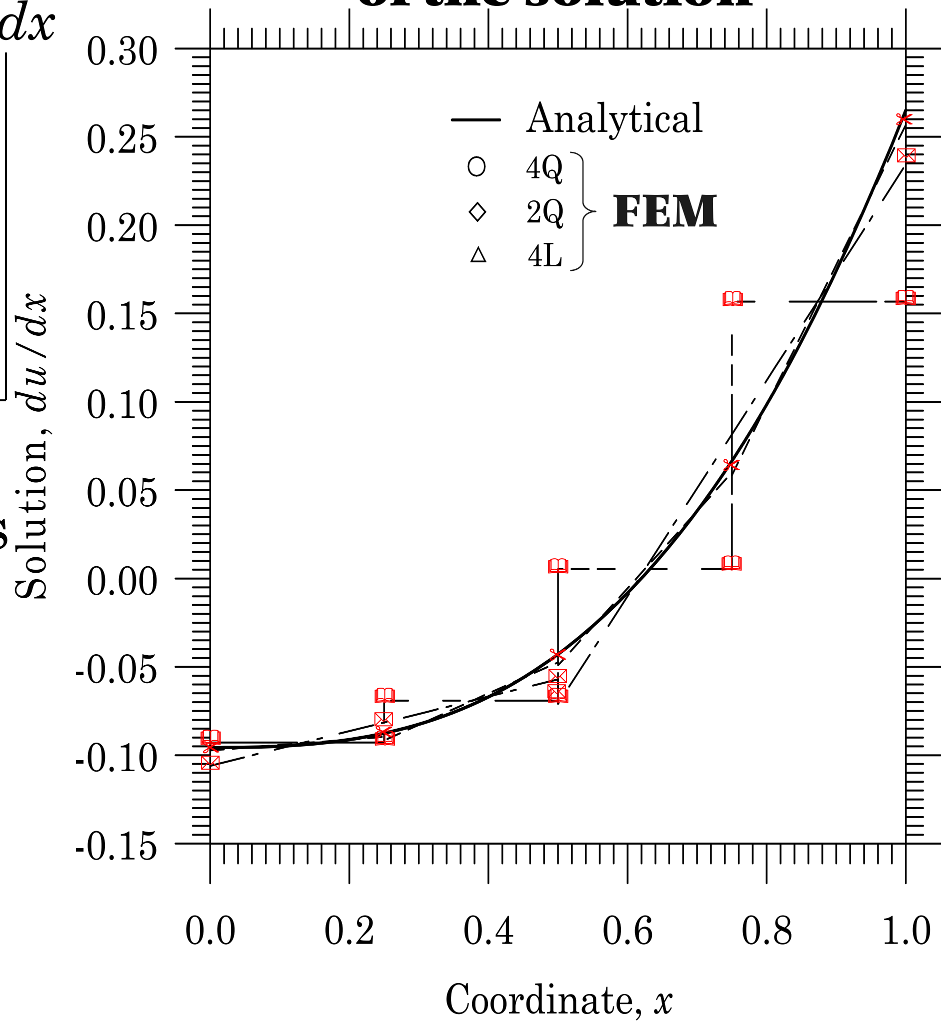
A DIFFERENTIAL EQUATION (cont.)

$$\frac{du_h^e}{dx} = \sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \neq \frac{du_h^{e+1}}{dx} = \sum_{j=1}^{n} u_j^{e+1} \frac{d\psi_j^{e+1}}{dx}$$

Same at the nodes common to elements

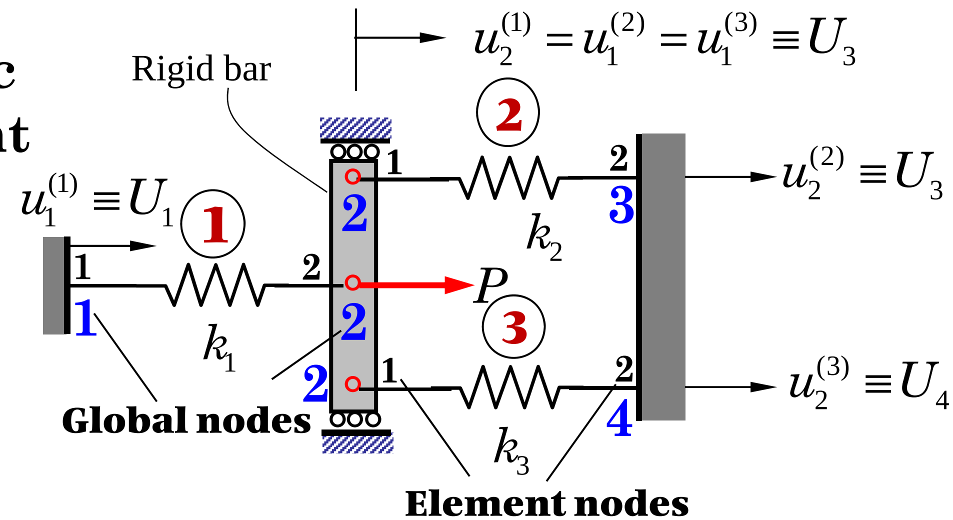
Not the same at the nodes common to elements

Plot of the derivative of the solution



A Network of Springs (No Des)

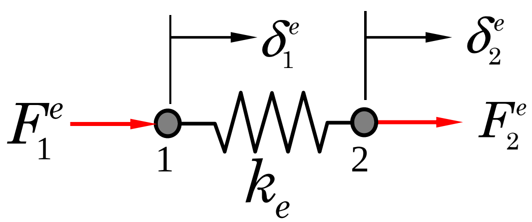
2 A network of linear elastic springs with finite element representation



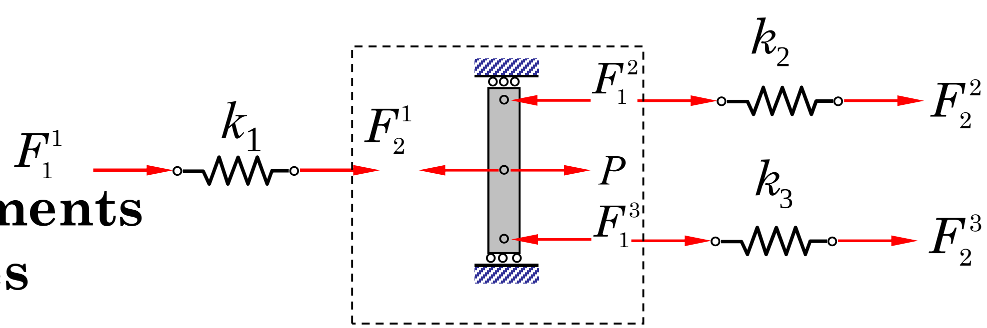
$$[\mathrm{B}] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$$

Typical spring element and its force-displacement relations

$$[K^e] = k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix}$$



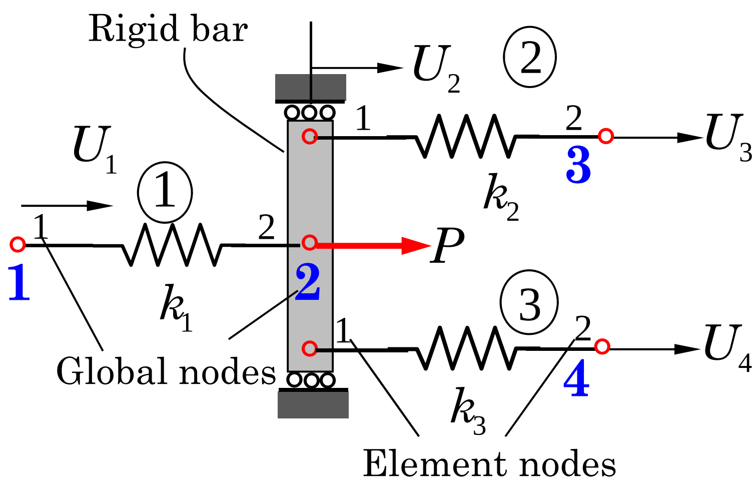
Individual elements and their forces



Balance of forces

$$P - F_2^1 - F_1^2 - F_1^3 = 0$$

Network of Springs (cont.)



$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$$

$$K_{11} = k_{11}^{(1)}, \;\; K_{12} = k_{12}^{(1)},$$
$$K_{22} = k_{22}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)},$$
$$K_{33} = k_{22}^{(2)}, \;\; K_{44} = k_{22}^{(3)},$$
$$K_{23} = k_{12}^{(2)}, \;\; K_{24} = k_{12}^{(3)}.$$

Network of Springs (Cont.)

$$k_1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^{(1)}\\ u_2^{(1)}\end{Bmatrix}=\begin{Bmatrix}F_1^{(1)}\\ F_2^{(1)}\end{Bmatrix} \quad (1)$$

$$k_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^{(2)}\\ u_2^{(2)}\end{Bmatrix}=\begin{Bmatrix}F_1^{(2)}\\ F_2^{(2)}\end{Bmatrix} \quad (2)\ (3)$$

$$k_3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^{(3)}\\ u_2^{(3)}\end{Bmatrix}=\begin{Bmatrix}F_1^{(3)}\\ F_2^{(3)}\end{Bmatrix} \quad (4)$$

$$P - F_2^{(1)} - F_1^{(2)} - F_1^{(3)} = 0$$

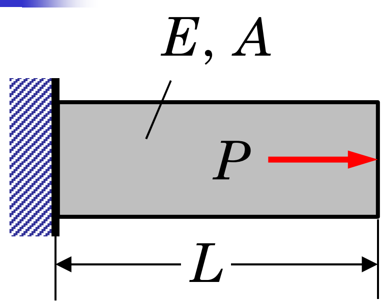
$$U_2 = \frac{P}{k_1 + k_2 + k_3}$$

$$F_1^{(1)} = -k_1 U_2,\ F_2^{(2)} = -k_2 U_2,\ F_2^{(3)} = -k_3 U_2$$

$$\begin{bmatrix}k_1 & -k_1 & 0 & 0\\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3\\ 0 & -k_2 & k_2 & 0\\ 0 & -k_3 & 0 & k_3\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}F_1^{(1)}\\ F_2^{(1)}+F_1^{(2)}+F_1^{(3)}\\ F_2^{(2)}\\ F_2^{(3)}\end{Bmatrix}$$

Examples of Uniaxially-Loaded Members

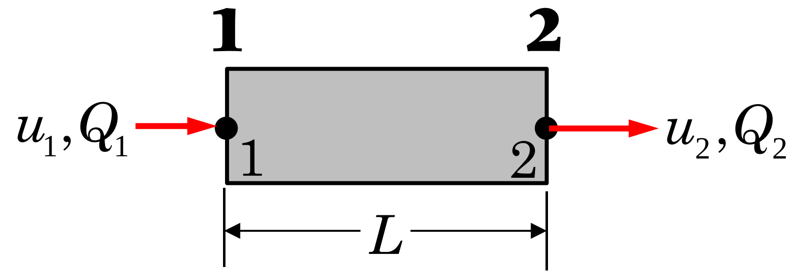
3



Problem

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=0$$

GDE

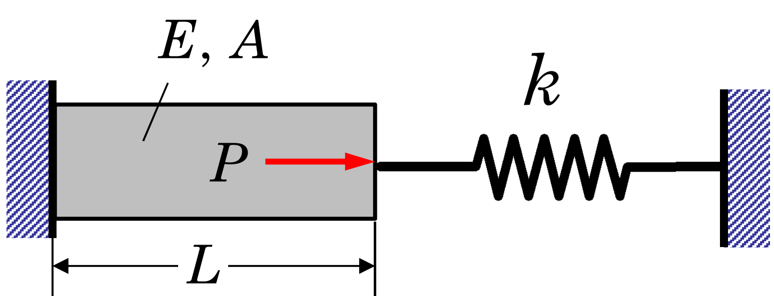


Finite element mesh

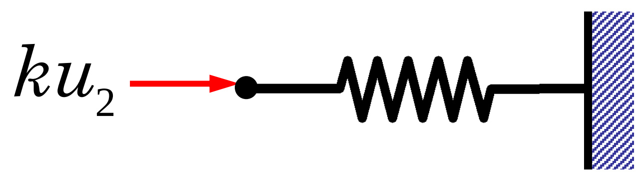
Solution:

$$\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\cancelto{0}{u_1}\\ u_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}+\begin{Bmatrix}Q_1\\ \cancelto{P}{Q_2}\end{Bmatrix} \Rightarrow u_2=\frac{PL}{EA},\quad Q_1=-\frac{EA}{L}u_2=-P$$

4



Problem

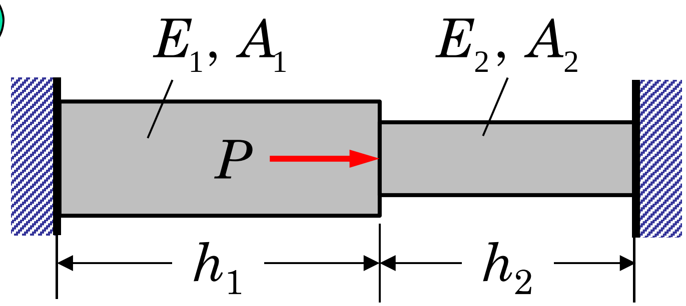


$$\boxed{Q_2 = P - ku_2}$$

Spring force

UNIAXIALLY-LOADED MEMBERS (cont.)

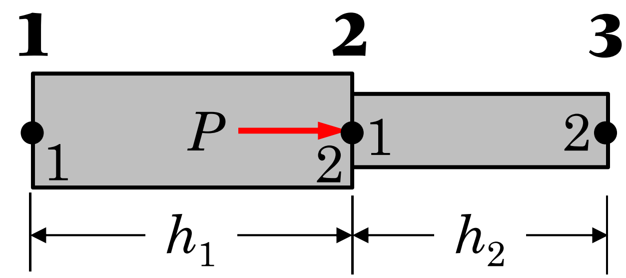
5



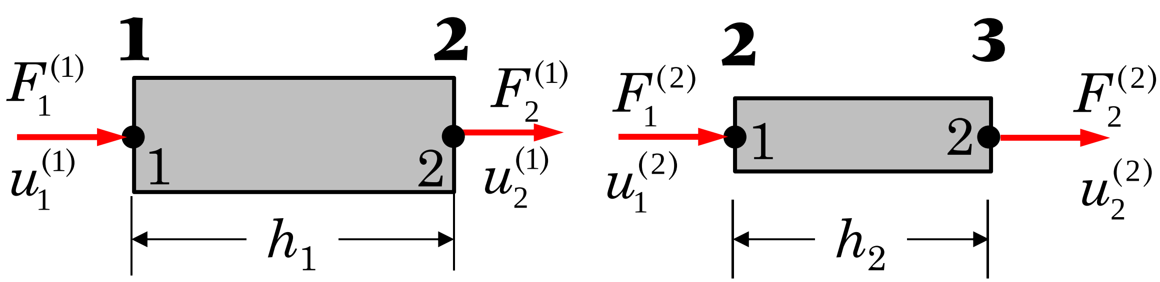
Problem

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=0$$

GDE



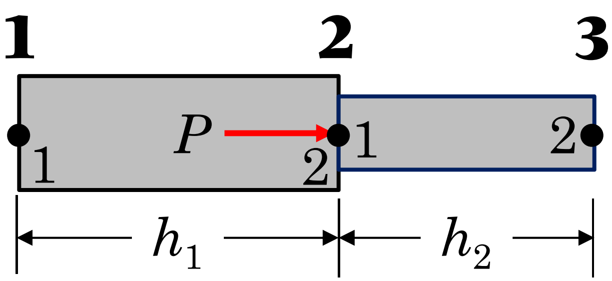
Finite element mesh



$$\begin{cases} F_2^{(1)} + F_1^{(2)} = P \\ u_1^{(1)} = U_1 \\ u_2^{(1)} = u_1^{(2)} = U_2 \\ u_2^{(2)} = U_3 \end{cases}$$

Inter-element compatibility

UNIAXIALLY-LOADED MEMBERS (cont.)



Finite element mesh

$$\begin{bmatrix} \frac{E_1A_1}{h_1} & -\frac{E_1A_1}{h_1} & 0 \\ -\frac{E_1A_1}{h_1} & \frac{E_1A_1}{h_1}+\frac{E_2A_2}{h_2} & -\frac{E_2A_2}{h_2} \\ 0 & -\frac{E_2A_2}{h_2} & \frac{E_2A_2}{h_2} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix}$$

Boundary conditions: $U_1 = U_3 = 0 \qquad F_2^{(1)} + F_1^{(2)} = P$

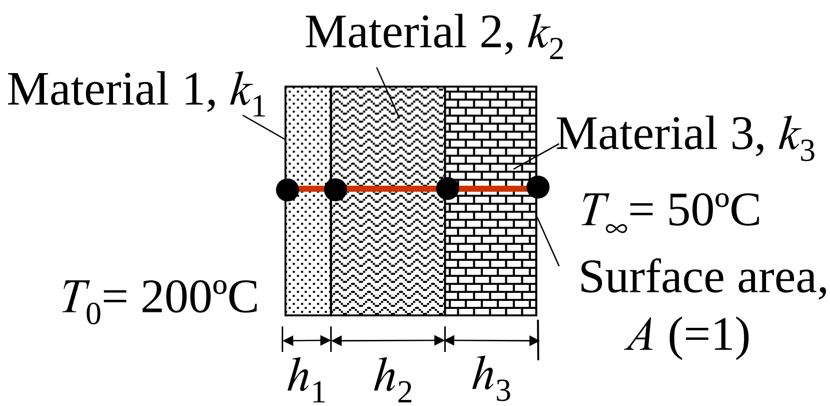
Solution:

$$U_2 = \frac{P}{\frac{E_1A_1}{h_1} + \frac{E_2A_2}{h_2}}$$

Post-computation:

$$F_1^{(1)} = -\frac{E_1A_1}{h_1}U_2, \quad F_2^{(2)} = -\frac{E_2A_2}{h_2}U_2$$

6 1D Heat flow through a composite wall



$$-\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0,\quad A=1$$

GDE

$$k\frac{dT}{dx}+\beta(T-T_\infty)=0$$

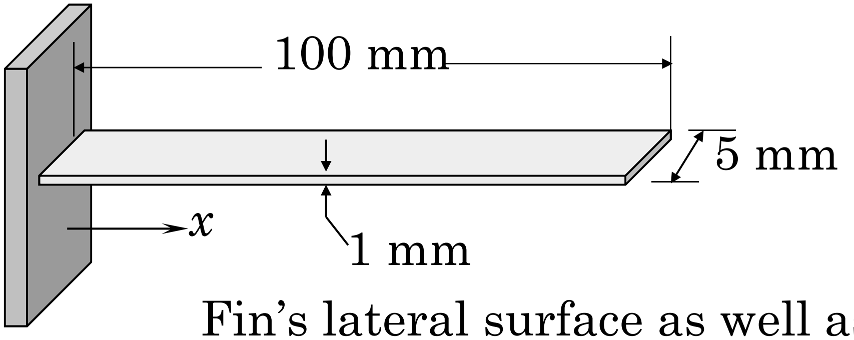
$$S_e \equiv \frac{k_e}{h_e}$$

$$[K^e]=s_e\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\cancel{\frac{c_e h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}},\quad \{F^e\}=\frac{f_e h_e}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}+\begin{Bmatrix}Q_1\\ Q_2\end{Bmatrix}$$

$$\begin{bmatrix} s_1 & -s_1 & 0 & 0\\ -s_1 & s_1+s_2 & -s_2 & 0\\ 0 & -s_2 & s_2+s_3 & -s_3\\ 0 & 0 & -s_3 & s_3 \end{bmatrix}\begin{Bmatrix}\cancelto{200}{U_1}\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}+\begin{Bmatrix}Q_1^{(1)}\\ \cancelto{0}{Q_2^{(1)}+Q_1^{(2)}}\\ \cancelto{0}{Q_2^{(2)}+Q_1^{(3)}}\\ \cancelto{-\beta(T_4-T_\infty)}{Q_2^{(3)}}\end{Bmatrix}$$

7 1D Heat flow through a fin with convection

Problem: Find the temperature distribution



$T(0) = T_0$

Fin's lateral surface as well as the end $x = L$ are exposed to ambient temperature, T_∞

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+P\beta(T-T_\infty)=0$$

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+P\beta T=P\beta T_\infty$$

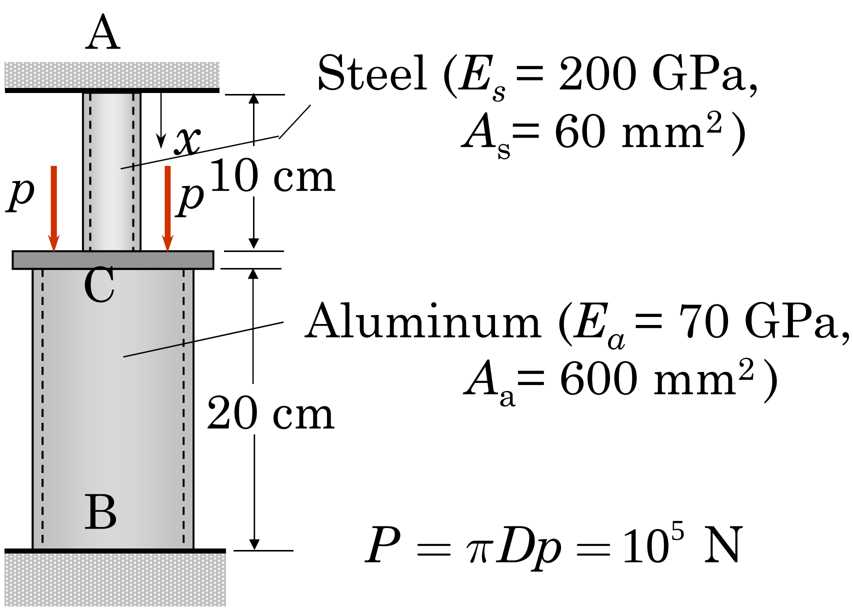
$$[K^e]=\frac{Ak}{h_e}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}+\frac{P\beta h_e}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\equiv\begin{bmatrix}s_1&s_2\\s_2&s_1\end{bmatrix},\quad \{F^e\}=\cancelto{f}{\frac{P\beta T_\infty h_e}{2}}\begin{Bmatrix}1\\1\end{Bmatrix}+\begin{Bmatrix}Q_1\\Q_2\end{Bmatrix}$$

$$Q_2^4 = k\frac{dT}{dx} = -\beta(T-T_\infty)$$

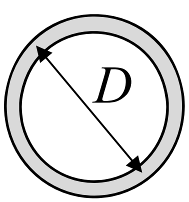
$$\begin{bmatrix}s_1&s_2&0&0&0\\s_2&2s_1&s_2&0&0\\0&s_2&2s_1&s_2&0\\0&0&s_2&2s_1&s_2\\0&0&0&s_2&s_1\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\U_3\\U_4\\U_5\end{Bmatrix}=\begin{Bmatrix}f\\f+f\\f+f\\f+f\\f\end{Bmatrix}+\begin{Bmatrix}Q_1^1\\\cancelto{0}{Q_2^1+Q_1^2}\\\cancelto{0}{Q_2^2+Q_1^3}\\\cancelto{0}{Q_2^3+Q_1^4}\\\cancelto{\text{known}}{Q_2^4}\end{Bmatrix}$$

EXERCISE PROBLEM FOR DISCUSSION

Problem: Find stresses in each member



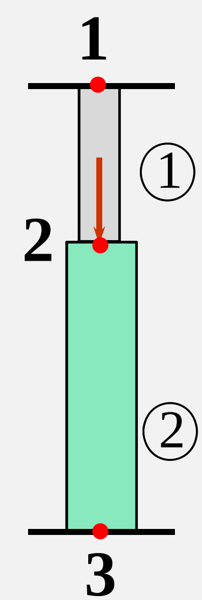
$$P = \pi D p = 10^5 \text{ N}$$



$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

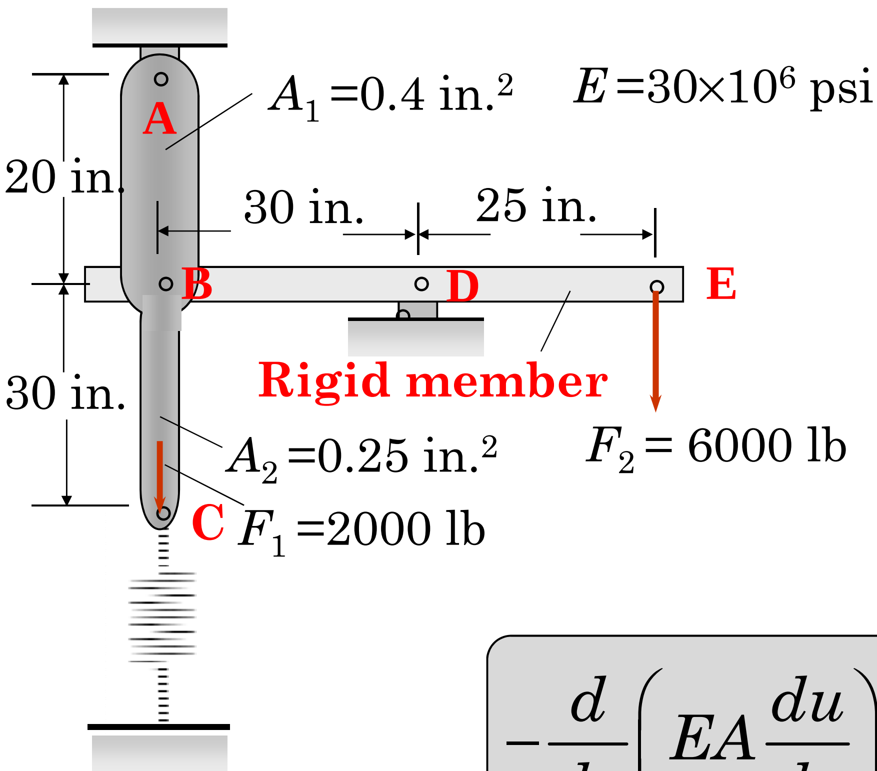
GDE

Problem set up and FEA



EXERCISE PROBLEM FOR DISCUSSION

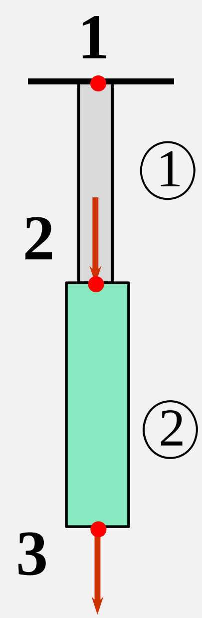
Problem: Find stresses in each member



$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

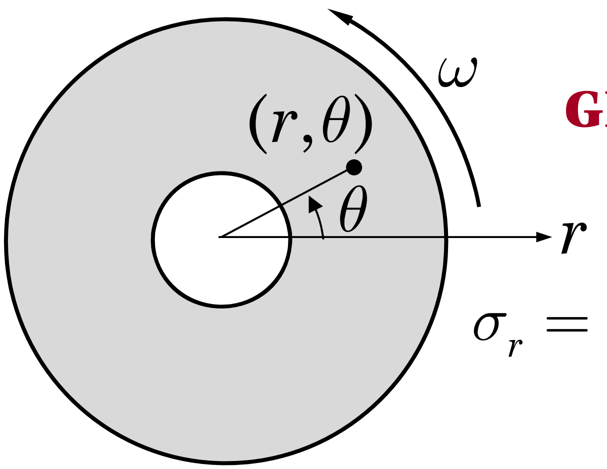
GDE

Problem set up and FEA



EXERCISE PROBLEM FOR DISCUSSION

Problem: Find the hoop stress in the rotating circular disc



GDE: $$-\frac{1}{r}\frac{d}{dr}(rt\sigma_r)+\frac{t\sigma_\theta}{r}=t\rho\omega^2 r,$$

$$\sigma_r = c\left(\frac{du}{dr}+\nu\frac{u}{r}\right),\quad \sigma_\theta = c\left(\frac{u}{r}+\nu\frac{du}{dr}\right),\quad c=\frac{E}{1-\nu^2}$$

Weak Form: $$f_0 = t\rho\omega^2 r,\, Q_a \equiv 2\pi(-tr\sigma_r)_a,\, Q_b \equiv 2\pi(tr\sigma_r)_b$$

$$0=\int_0^{2\pi}\int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}(tr\sigma_r)+\frac{t\sigma_\theta}{r}-f_0\right] r\,dr\,d\theta$$

$$=2\pi\int_{r_a}^{r_b}\left(tr\frac{dw}{dr}\sigma_r + wt\sigma_\theta - wf_0 r\right)dr - Q_a w(r_a) - Q_b w(r_b)$$

EXERCISE PROBLEM FOR DISCUSSION: spinning disc problem

(continued)

Finite Element Model:

$$[K^e]\{u^e\} = \{F^e\} \quad \text{or} \quad \mathbf{K}^e\mathbf{u}^e = \mathbf{F}^e$$

$$K_{ij}^e = 2\pi \int_{r_a}^{r_b} ct \left[r \frac{d\psi_i}{dr} \left(\frac{d\psi_j}{dr} + \frac{\nu}{r}\psi_j \right) + \psi_i \left(\frac{1}{r}\psi_j + \nu \frac{d\psi_j}{dr} \right) \right] dr$$

$$F_i^e = 2\pi \int_{r_a}^{r_b} f_0 \psi_i \, r dr + Q_a \psi_i(r_a) + Q_b \psi_i(r_b)$$

$$f_0 = t\rho\omega^2 r, Q_a \equiv 2\pi(-tr\sigma_r)_a, Q_b \equiv 2\pi(tr\sigma_r)_b$$

SUMMARY

Beginning with a model second-order differential equation that arises, for example, in connection with axial deformation of bars, 1-D heat transfer in fins of a heat exchanger, or 1-D flow through pipes and channels, the following steps are used to in the finite element analysis of the problem:

1. **Divided the domain into subdomains, called *finite elements*.**

2. **Over each element, an integral statement, called *weak form*, is developed. The weak form is equivalent to the differential equation as well as specified natural boundary conditions on the boundary of the element.**

SUMMARY (continued)

3. Using polynomial approximation of the variables, a system of algebraic equations, called **finite element model**, is developed. The model relates the nodal values of the PVs and the SVs.

4. The element equations are then **assembled** to eliminate excessive unknown SVs by requiring continuity of PVs and balance of SVs at the nodes.

5. The assembled system of equations are then **solved** for the unknown PVs at the nodes by using the known boundary conditions.

6. Post-computation may be used to compute SVs and PVs at points other than nodes. **The SVs are discontinuous between elements.**