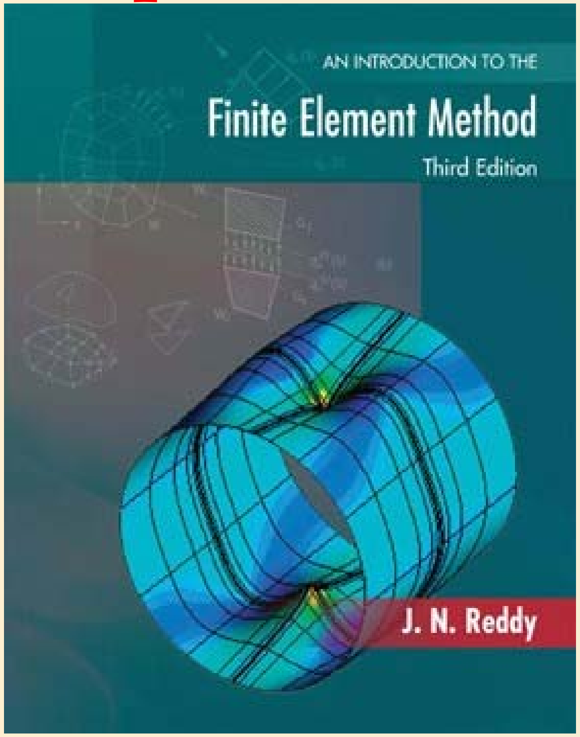
The Finite Element Method

GENERAL INTRODUCTION

Read:

Chapters 1 and 2



CONTENTS

- Engineering and analysis
- Simulation of a physical process
- Examples mathematical model development
- Approximate solutions and methods of approximation
- The basic features of the finite element method
- Examples
- Finite element discretization
- Terminology
- Steps involved in the finite element model development

INTRODUCTORY REMARKS

➢ WHAT WE DO AS ENGINEERS?

- develop mathematical models,
- conduct physical experiments,
- carry out numerical simulations to help designer, and
- design and build systems to achieve a (1) *functionality* in (2) *most economical way*.

Knowing the fundamentals associated with each engineering problem you set out to tackle, not only makes you a better engineer but also empowers you as an engineer.

INTRODUCTORY REMARKS

<u>Engineering</u> is the discipline, art, and profession of acquiring and applying technical, scientific, and mathematical knowledge to design and implement materials, structures, machines, devices, systems, and <u>processes</u> that *safely realize a desired objective*.

Engineering is a problem-solving discipline, and solution requires an understanding of the phenomena that occurs in the system or process.

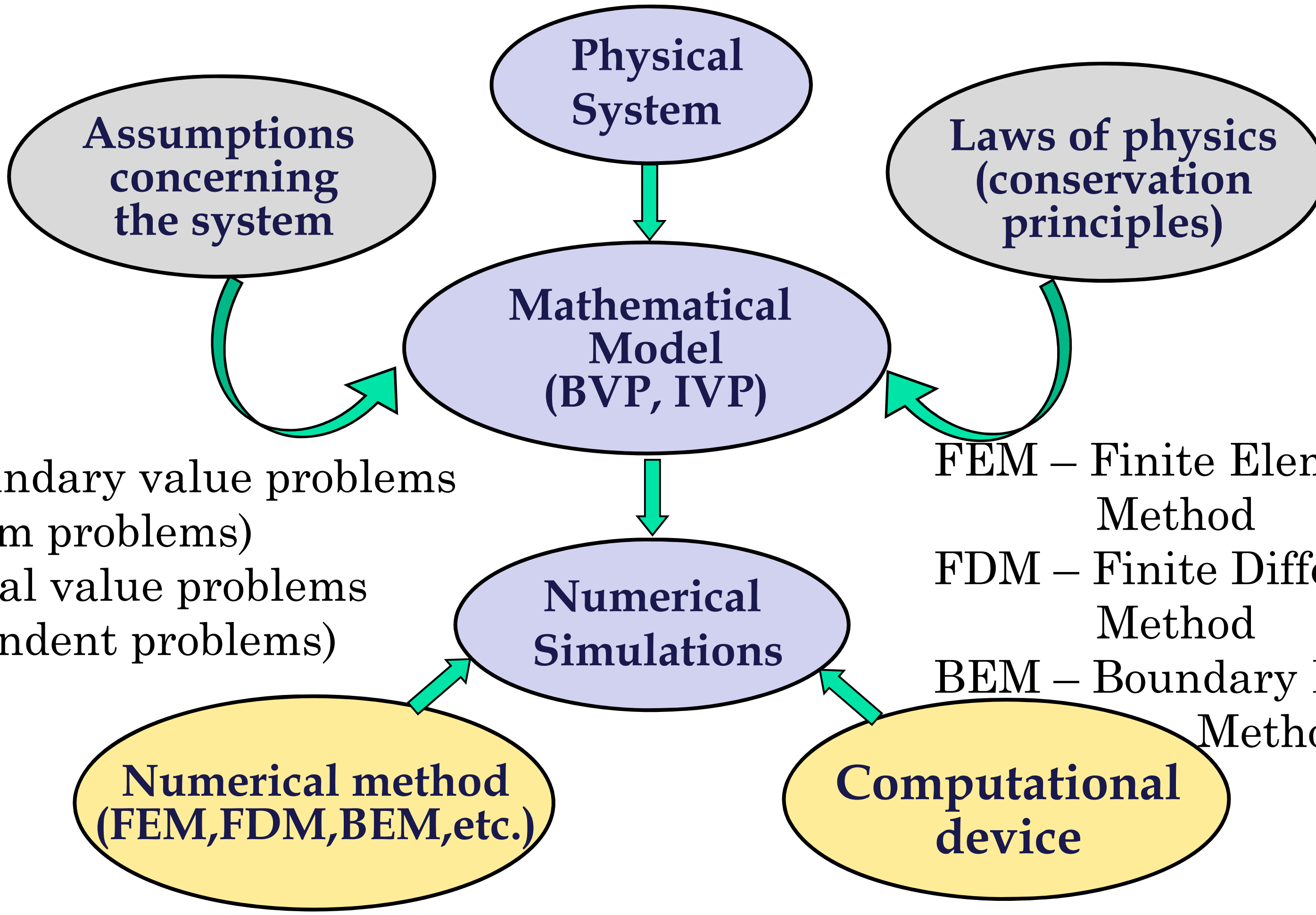
Analysis

Analysis **is an aid to design and manufacturing, and not an end in itself.**

<u>Analysis involves the following steps</u>:

- **identifying the <u>problem</u> and nature of the <u>response</u> to be determined,**
- **selecting the <u>mathematical model</u> (i.e., governing equations),**
- **solving the problem with a <u>solution method (e.g., FEM)</u>, and**
- **<u>evaluating the results</u> in light of the design parameters.**

MODELING OF A Physical Process

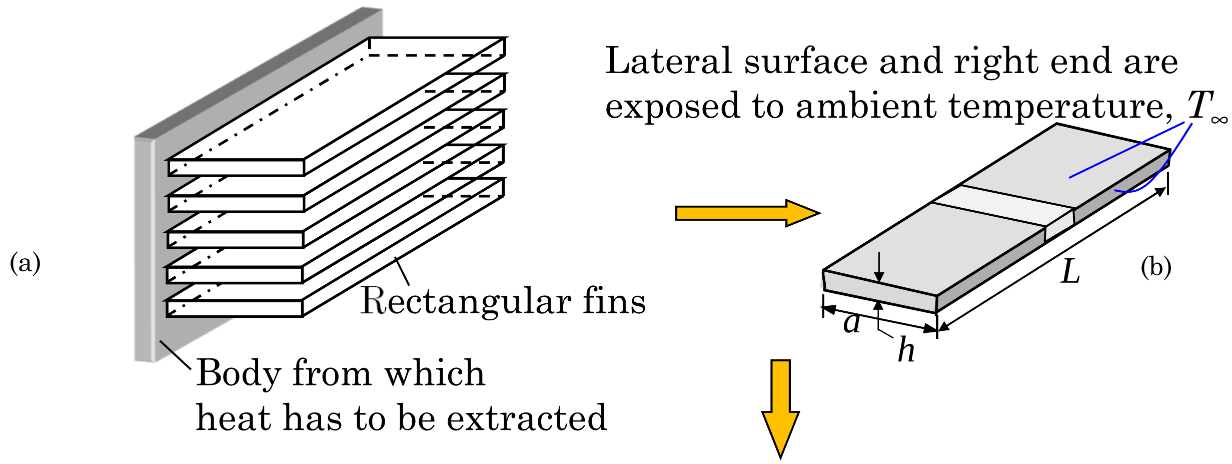


BVP – Boundary value problems (equilibrium problems)
IVP – Initial value problems (time-dependent problems)

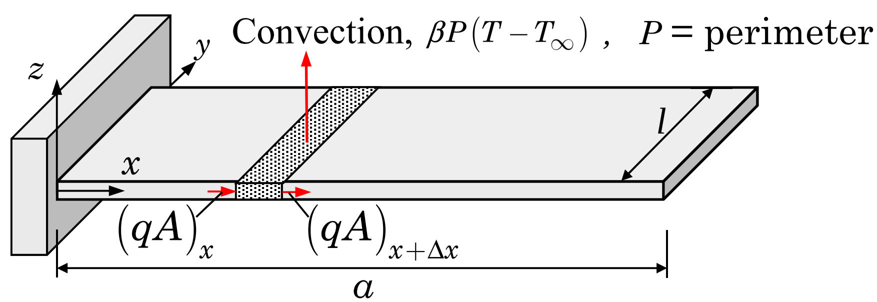
FEM – Finite Element Method
FDM – Finite Difference Method
BEM – Boundary Element Method

EXAMPLES OF MATHEMATICAL MODEL DEVELOPMENT

Objective: Determine heat flow in a heat exchanger fin



3D to 2D



2D to 1D

$$(qA)_x - (qA)_{x+\Delta x} - \beta P \Delta x (T - T_\infty) + \rho r_h \left(\frac{A_x + A_{x+\Delta x}}{2} \right) \Delta x = 0$$

$$-\frac{d}{dx}(Aq) - \beta P (T - T_\infty) + \rho r_h A = 0, \quad q = -k \frac{dT}{dx}$$

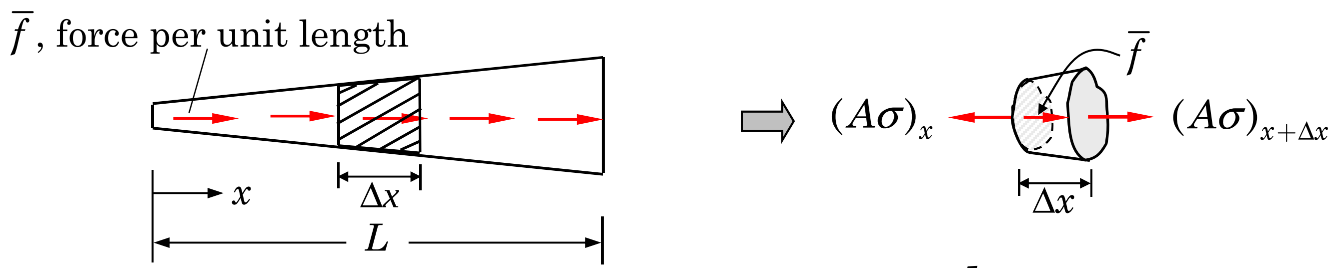
EXAMPLE OF ENGINEERING MODEL DEVELOPMENT (continued)

$$-\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+\beta P(T-T_\infty)=\rho r_h A$$

$$u=T-T_\infty,\quad a=Ak,\quad c=\beta P,\quad \rho r_h A=f$$

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f$$

Determine: Axial deformation of a bar



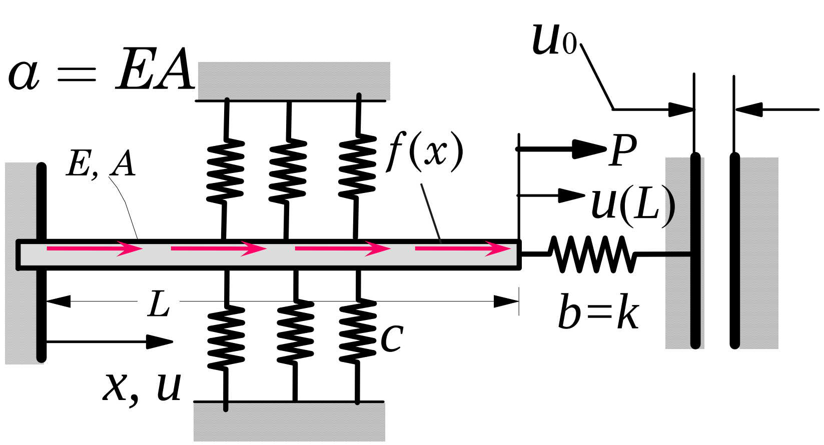
$$(A\sigma)_{x+\Delta x}-(A\sigma)_x+\bar{f}\ A\ \Delta x=0\quad\Rightarrow\quad \frac{d}{dx}(A\sigma)+\bar{f}\ A=0,$$

$$\sigma=E\varepsilon,\quad \varepsilon=\frac{du}{dx}\ ,\qquad -\frac{d}{dx}\left(AE\frac{du}{dx}\right)+f\ =0$$

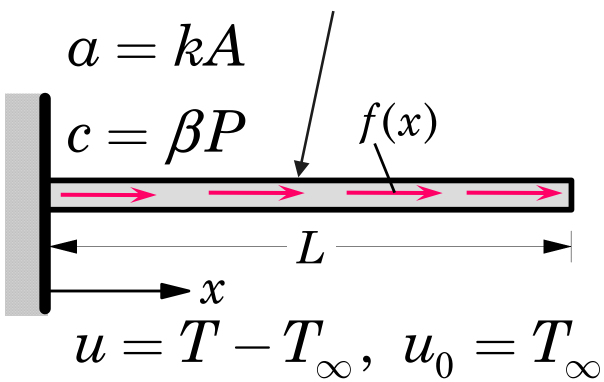
Model Problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)=0 \text{ in } \Omega=(0,L)$$

$$a\frac{du}{dx}+b(u-u_0)=P \text{ at a boundary point}$$



Elastic deformation of a bar



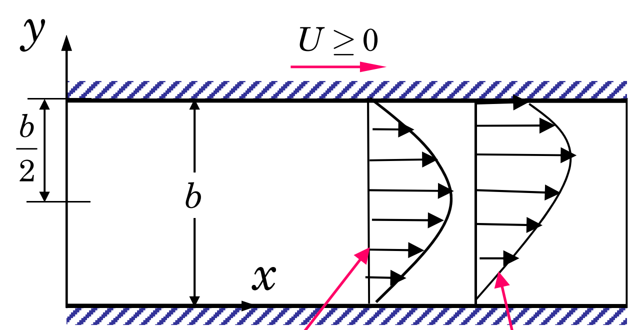
Heat transfer in a fin

ENGINEERING EXAMPLES OF THE MODEL PROBLEM IN 1-D

Flow of viscous fluid through a channel

$$-\frac{d}{dy}\left(\mu \frac{dv_x}{dy}\right) - f = 0 \text{ in } \Omega = (0,b)$$

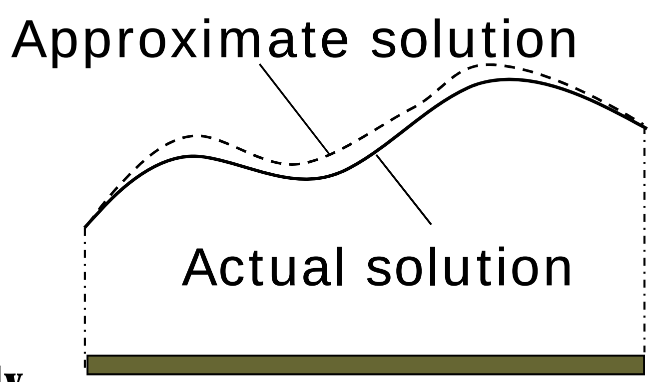
$u(y) = v_x(y)$, horizontal velocity
f = pressure gradient, dp/dx (constant)
$\mu(y)$ = fluid viscosity
Q = shear stress
U = velocity of the top surface



Exact and Approximate Solutions

An *exact solution* satisfies (a) the differential equation at every point of the domain and (b) boundary conditions on the boundary. An *approximate solution* satisfies the differential equation as well as the specified boundary conditions in some "acceptable sense" (to be made clearer shortly). We seek the approximate solution as a linear combination of unknown parameters c_i and known functions ϕ_i and ϕ_0

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} c_i \phi_i(x) + \phi_0(x)$$



Approximation of the actual solution over the entire domain

Determining Approximate Solutions (continued)

1. Suppose that ϕ_i is selected to satisfy the boundary conditions exactly. Then substitution of $u_N(x)$ into the differential equation

$$-\frac{d}{dx}\left(a(x)\frac{du_N}{dx}\right)+c(x)u_N-f(x)=0$$

will result in a non-zero function on the left side of the equality:

$$-\frac{d}{dx}\left(a(x)\frac{du_N}{dx}\right)+c(x)u_N-f(x)\equiv R(x)\neq 0$$

Then c_i are determined such that the residual, $R(x)$, is zero in some sense.

METHODS OF APPROXIMATION

1. One sense in which the residual, $R(x)$, can be made zero is to require it to be zero at selected number of points. The number of points should be equal to the number of unknowns in the approximate solution

$$u(x) \approx u_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$$

$\phi_j(x)$ and $\phi_0(x)$ are functions to be selected to satisfy the specified boundary conditions and c_j are parameters to be determined such that the residual is *made* zero in some sense.

This way of determining c_i is known as the *Collocation method*. We obtain N algebraic equations in N unknown C's

$$R(x_i) = 0, \quad i = 1, 2, \cdots, N$$

Methods of Approximation (Continued)

2. Another approach in which the residual, $R(x)$, can be made zero is in a least-squares sense; i.e., minimize the integral of the square of the residual with respect to C's.

$$\text{Minimize } J(c_1, c_2, \cdots, c_N) = \int_0^L R^2 \, dx$$

$$\text{or } \frac{\partial J}{\partial c_i} = 2\int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

This method is known as the *least-Squares method*. We obtain N algebraic equations in N unknown C's

$$\int_0^L R \frac{\partial R}{\partial c_i} dx = 0$$

Methods of Approximation (Continued)

3. Yet, another approach in which the residual, $R(x)$, can be made zero is in a weighted-residual sense

$$0 = \int_0^L \psi_i R\, dx,\ \ i = 1, 2, \cdots, N$$

where ψ_i are linearly independent set of functions

This method is known as the *Weighted-Residual method*. We obtain N algebraic equations in N unknown C's. In general, weight functions ψ_i are not the same as the approximation functions φ_i. Various special cases are

Petrov-Galerkin Method: $\psi_i \neq \phi_i$

Galerkin Method: $\psi_i = \phi_i$

WEIGHTED-INTEGRAL FORMULATIONS
in the Numerical Solution of Differential Eqs.

The approximation methods discussed earlier can be viewed as special cases of the weighted-residual methods of approximation. In particular, we have

- Collocation method $\psi_i(x) = \delta(x - x_i)$
- Least-squares method $\psi_i(x) = \dfrac{\partial R}{\partial c_i}$
- ♦ Petrov-Galerkin method $\psi_i(x) \neq \phi_i(x)$
- ♦ Galerkin Method $\psi_i(x) = \phi_i(x)$

Methods of Approximation (Continued)

4. Another approach in which the governing equation is cast in a **weak-form** and the weight function is taken the same as the approximation function is known as the *Ritz method*:

$$B(\phi_i, u_N) = \ell(\phi_i), \;\; i = 1, 2, \ldots, N$$

The Ritz method is the most commonly used method for all commercial software. In this method, ϕ_0 satisfies only the specified essential (geometric) boundary conditions while ϕ_i satisfies the homogeneous form of the specified essential boundary conditions. The specified natural (force) boundary condition are included in the weak form.

WORKING EXAMPLE

For a weighted-residual method:

ϕ_0 satisfy the actual specified BCs

ϕ_i satisfy the homogeneous form of the actual specified BCs

For the Ritz method:

ϕ_0 satisfy the actual specified <u>essential</u> BCs

ϕ_i satisfy the homogeneous form of the actual specified <u>essential</u> BCs

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f_0=0,\quad 0<x<L$$

$$u(0)=u_0,\;\; a\frac{du}{dx}\bigg|_{x=L}=P$$

WORKING EXAMPLE
(CONTINUED)

For a weighted-residual method:

$$\phi_0(0) = u_0, \quad a\frac{d\phi_0}{dx}\bigg|_{x=L} = P \Rightarrow \phi_0 = u_0 + \frac{P}{a}x$$

$$\phi_i(0) = 0, \quad a\frac{d\phi_i}{dx}\bigg|_{x=L} = 0 \Rightarrow \phi_1 = x(2L - x)$$

For the Ritz (or weak-form Galerkin) method:

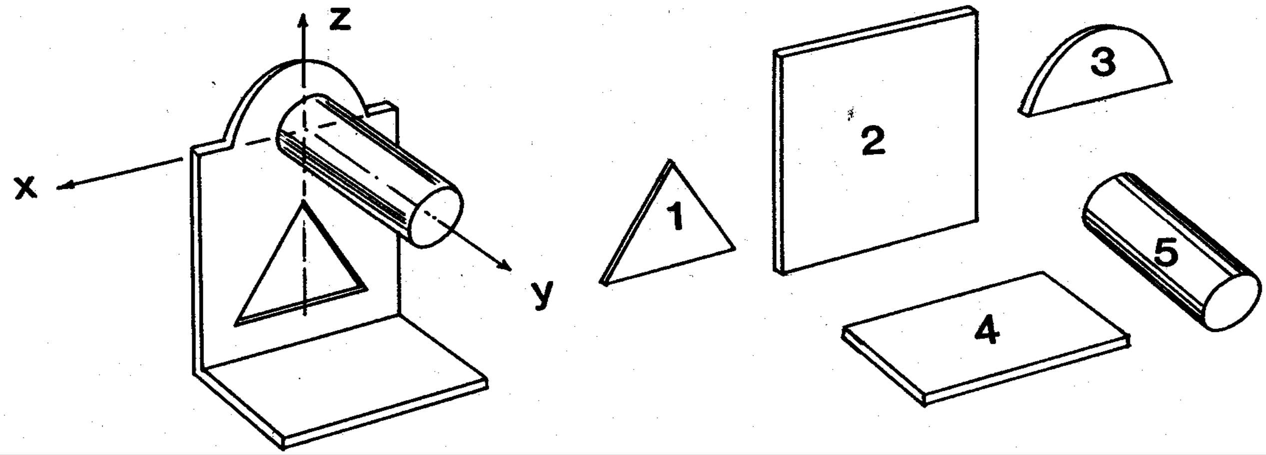
$$\phi_0(0) = u_0, \quad \phi_i(0) = 0$$

$$\Rightarrow \quad \phi_0 = u_0, \quad \phi_i(x) = x^i$$

BASIC FEATURES OF THE FINITE ELEMENT METHOD (FEM)

- Divide whole into parts (*finite element mesh*)
- Set up the `problem' over a typical part
  (derive a set of relationships between primary and secondary variables)
- Assemble the parts to obtain the solution to the whole

Example 1: Determine the center of mass of a 3D machine part

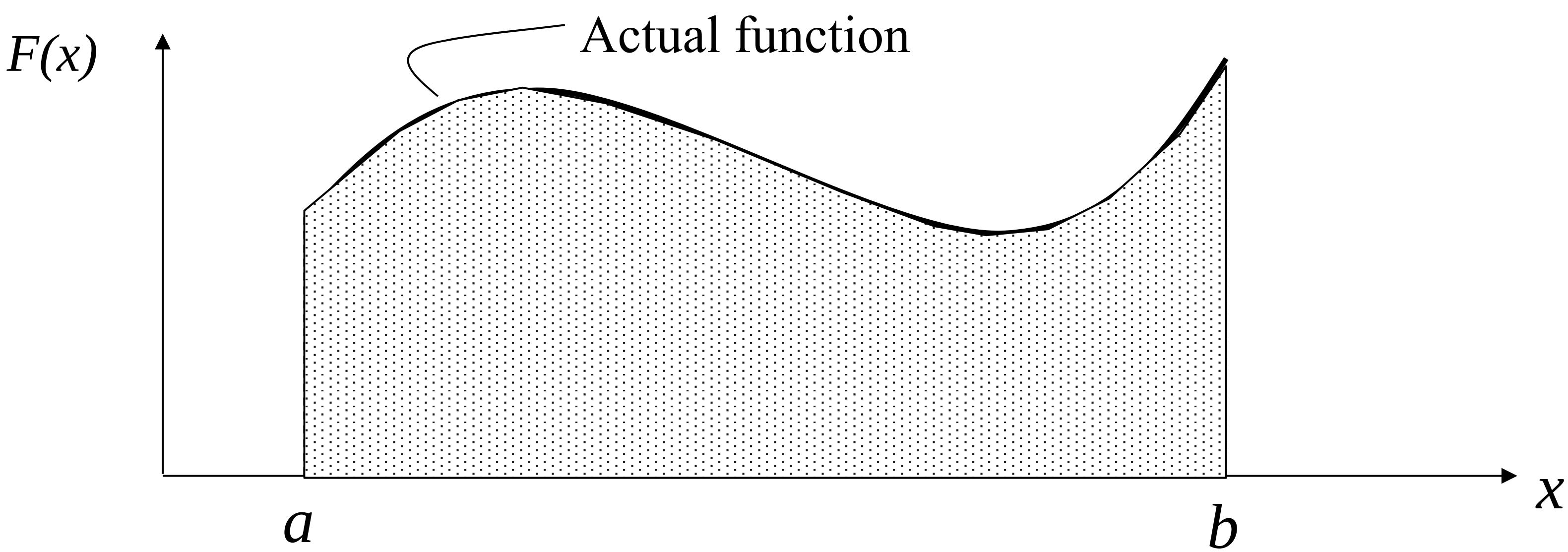


1. See the simplicity in the complicated (see the geometric shapes that are simple to identify the center of mass).
2. Determine the center of mass of each part.
3. Put the parts together to obtain the required solution.

$$X = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}, \quad Y = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i}, \quad Z = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i}$$

Example 2: Determine the integral of a function

$$I = \int_a^b F(x)\,dx$$

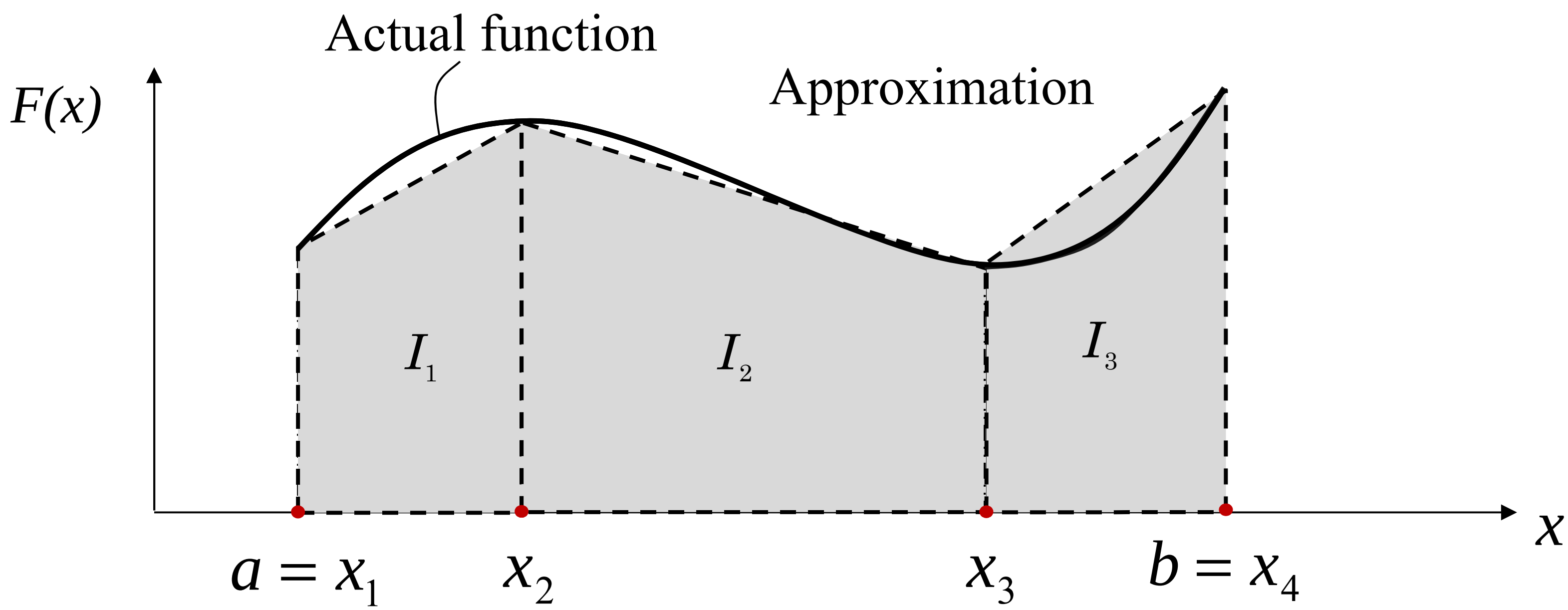


EXAMPLE 2 (continued)

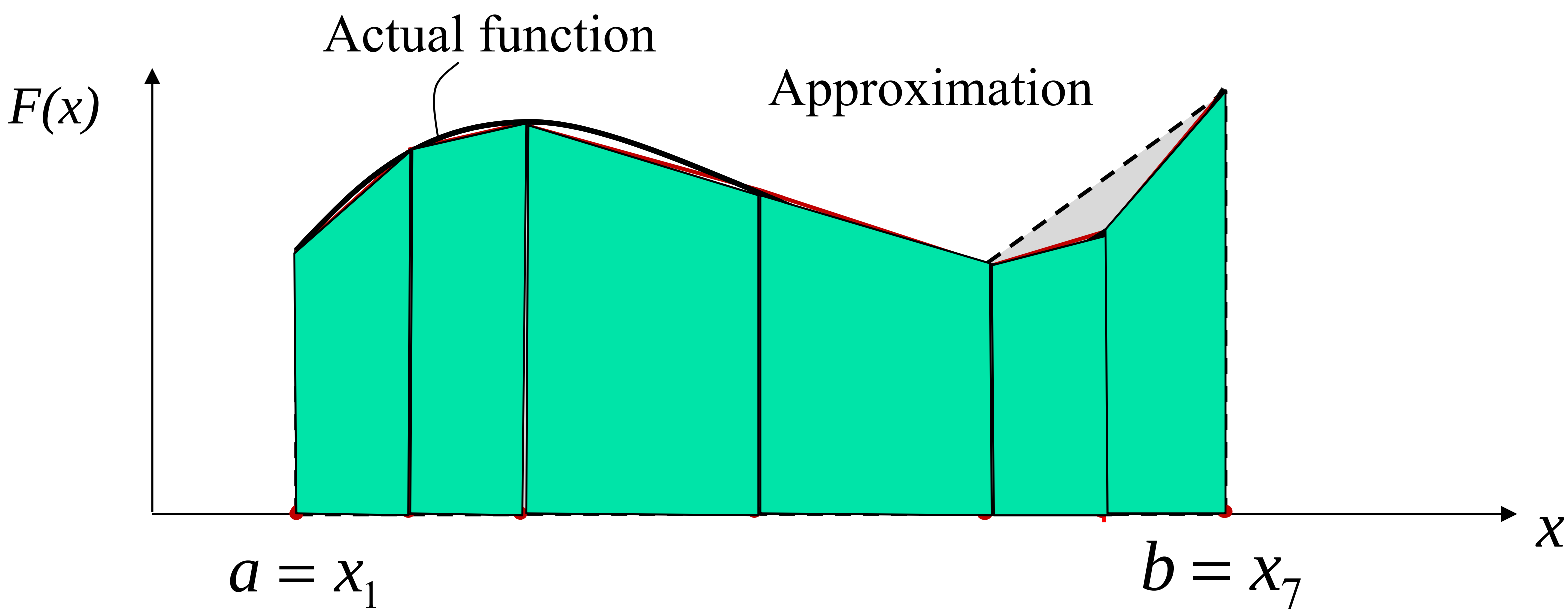
$$F(x) \approx \begin{cases} a_1 + b_1 x, & a \le x \le x_2 \\ a_2 + b_2 x, & x_2 \le x \le x_3 \\ a_3 + b_3 x, & x_3 \le x \le b \end{cases}$$

$$I_i = \int_{x_i}^{x_{i+1}} F_i(x)\,dx = \int_{x_i}^{x_{i+1}} (a_i + b_i x)\,dx$$

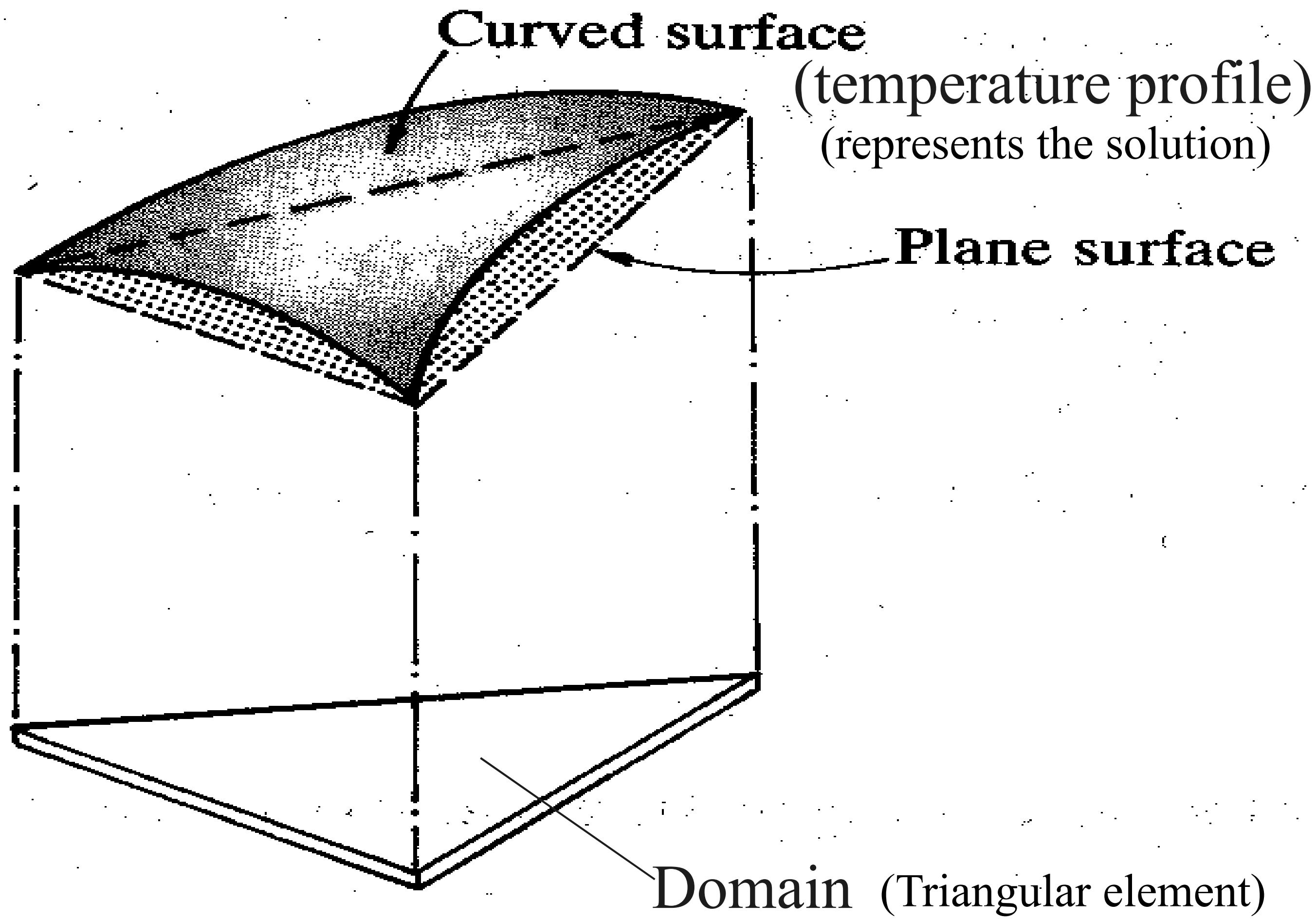
$$I \approx I_1 + I_2 + I_3$$



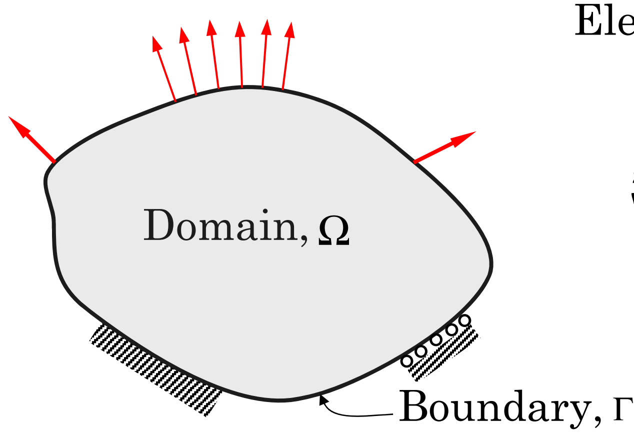
EXAMPLE 2 (continued) – Refined mesh



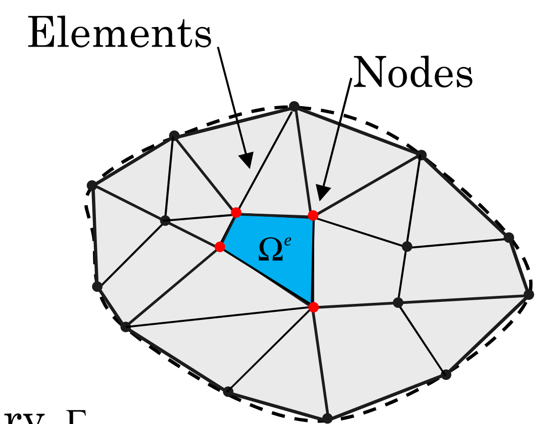
Approximation of a curved surface with a plane



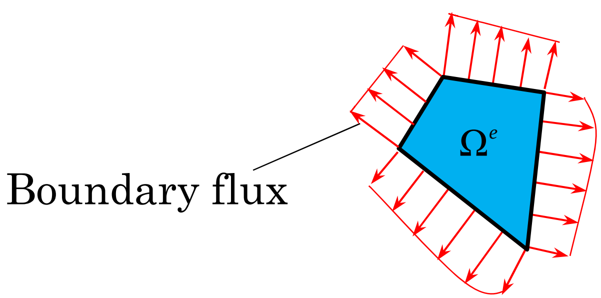
Finite Element Discretization



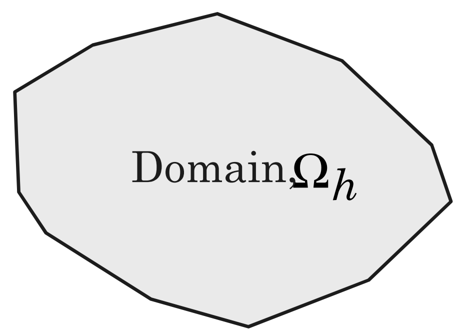
(*a*) Given domain



(*b*) Finite element mesh

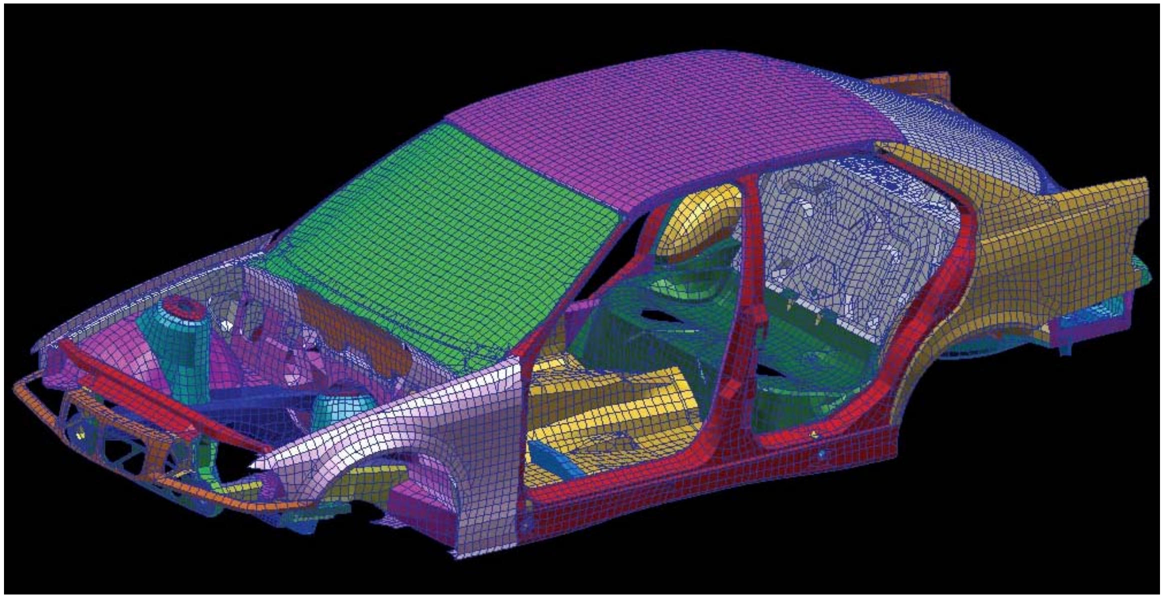


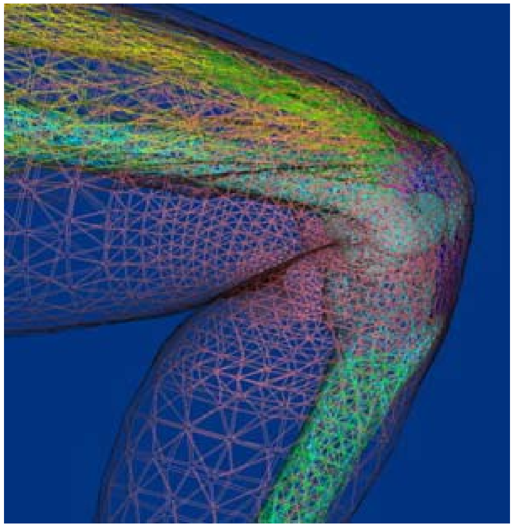
(*c*) Typical element with boundary fluxes

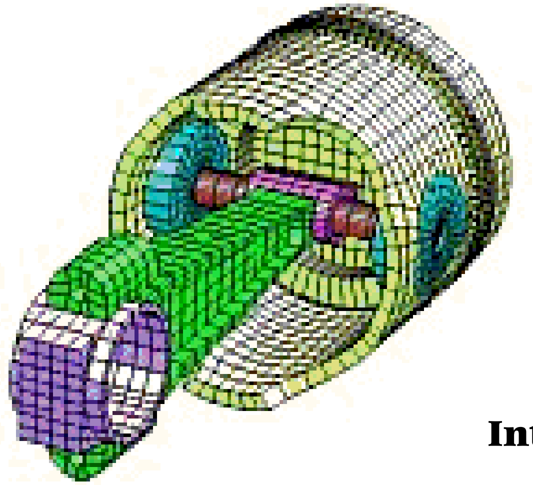


(*d*) Discretized domain

Some Examples of Real-World Finite Element Discretizations







FEM Terminology

- ***Element*** A geometric sub-domain of the region being simulated, with the property that it allows a unique (1) representation of its geometry and (2) derivation of the approximation (interpolation) functions.
- ***Node*** A geometric location in the element which plays a role in the derivation of the interpolation functions and it is the point at which solution is sought.
- ***Mesh*** A collection of elements (or nodes) that replaces the actual domain.
- ***Weak Form*** An integral statement equivalent to the governing equations and *natural* boundary conditions. **More to come.**

FEM Terminology (continued)

- ***Finite Element Model*** A set of algebraic equations relating the nodal values of the primary variables (e.g., displacements) to the nodal values of the secondary variables (e.g., forces) in an element.
- ***Finite element model*** is NOT the same as the *finite element method.* There is only one finite element method but there can be more than one finite element model of a problem (depending on the approximate method used to derive the algebraic equations).
- ***Numerical Simulation*** Evaluation of the mathematical model (i.e., solution of the governing equations) using a numerical method and computer.

Major Steps of Finite Element Model Development

- Begin with the *governing equations* of the problem
- Develop its ***weak form*** over a typical finite element
- *Approximate* the solution over each finite element
- Obtain algebraic relations among the *quantities of interest* over each finite element (i.e., <u>finite element model</u>)

Major Steps of Finite Element Model Development