

# **Shear Deformable Beams and Plates**

*Relationships with Classical Solutions*

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# Contents

<b>Preface</b> .....	xiii
<b>1 Introduction</b> .....	1
1.1 Preliminary Comments .....	1
1.2 An Overview of Plate Theories .....	3
1.3 Present Study .....	5
Problems .....	6
 <i>Part 1: Beams</i>	
<b>2 Bending of Beams</b> .....	11
2.1 Beam Theories .....	11
2.1.1 Introduction .....	11
2.1.2 Euler–Bernoulli Beam Theory (EBT) .....	14
2.1.3 Timoshenko Beam Theory (TBT) .....	17
2.1.4 Reddy–Bickford Beam Theory (RBT) .....	20
2.2 Relationships Between EBT and TBT .....	24
2.2.1 General Comments .....	24
2.2.2 Simply Supported (SS) Beams .....	25
2.2.3 Clamped-Free (CF) Beams .....	25
2.2.4 Free-Clamped (FC) Beams .....	25
2.2.5 Clamped-Simply Supported (CS) Beams .....	26
2.2.6 Simply Supported-Clamped (SC) Beams .....	26
2.2.7 Clamped (CC) Beams .....	26
2.2.8 Summary of Relationships .....	27
2.3 Relationships Between EBT and RBT .....	28

2.4 Examples . . . . .	31
2.4.1 Simply Supported Beam . . . . .	32
2.4.2 Cantilever Beam . . . . .	34
2.5 Summary . . . . .	36
Problems . . . . .	37
<b>3 Shear-Flexural Stiffness Matrix . . . . .</b>	<b>39</b>
3.1 Introduction . . . . .	39
3.2 Summary of Relationships . . . . .	41
3.2.1 Relationships Between TBT and EBT . . . . .	41
3.2.2 Relationships Between RBT and EBT . . . . .	41
3.2.3 Relationships Between Simplified RBT and EBT . . . . .	42
3.3 Stiffness Matrix . . . . .	44
3.4 Frame Structure - An Example . . . . .	48
3.5 Concluding Remarks . . . . .	50
Problems . . . . .	52
<b>4 Buckling of Columns . . . . .</b>	<b>55</b>
4.1 Introduction . . . . .	55
4.2 Relationship Between Euler–Bernoulli . . . . . and Timoshenko Columns . . . . .	56
4.2.1 General Relationship . . . . .	56
4.2.2 Pinned-Pinned Columns . . . . .	58
4.2.3 Fixed-Fixed Columns . . . . .	60
4.2.4 Fixed-Free Columns . . . . .	61
4.3 Relationship Between Euler–Bernoulli and . . . . . Reddy–Bickford Columns . . . . .	64
4.3.1 General Relationship . . . . .	64
4.3.2 Pinned-Pinned Columns . . . . .	69
4.3.3 Fixed-Fixed Columns . . . . .	70
4.3.4 Fixed-Free Columns . . . . .	71
4.3.5 Pinned-Pinned Columns with End Rotational . . . . . Springs of Equal Stiffness . . . . .	72
4.4 Concluding Remarks . . . . .	74
Problems . . . . .	75

<b>5 Tapered Beams</b> .....	77
5.1 Introduction .....	77
5.2 Stress Resultant-Displacement Relations .....	78
5.3 Equilibrium Equations .....	78
5.4 Deflection and Force Relationships .....	79
5.4.1 General Relationships .....	79
5.4.2 Simply Supported (SS) Beams .....	80
5.4.3 Clamped-Free (CF) Beams .....	81
5.4.4 Free-Clamped (FC) Beams .....	81
5.4.5 Clamped (CC) Beams .....	82
5.4.6 Clamped-Simply Supported (CS) Beams .....	82
5.4.7 Simply Supported-Clamped (SC) Beams .....	83
5.4.8 An Example .....	84
5.5 Symmetrically Laminated Beams .....	85
5.6 Concluding Remarks .....	86
Problems .....	86

*Part 2: Plates*

<b>6 Theories of Plate Bending</b> .....	89
6.1 Overview of Plate Theories .....	89
6.2 Classical (Kirchhoff) Plate Theory (CPT) .....	92
6.2.1 Equations of Equilibrium .....	92
6.2.2 Boundary Conditions .....	94
6.2.3 Governing Equations in Terms of the Deflection .....	98
6.3 First-Order Shear Deformation Plate Theory (FSDT) .....	100
6.3.1 Equations of Equilibrium .....	100
6.3.2 Plate Constitutive Equations .....	102
6.3.3 Governing Equations in Terms of Displacements .....	103
6.4 Third-Order Shear Deformation Plate Theory (TSDT) .....	105
6.4.1 Equations of Equilibrium .....	105
6.4.2 Plate Constitutive Equations .....	107
Problems .....	108

<b>7</b>	<b>Bending Relationships for Simply Supported Plates . . .</b>	<b>111</b>
7.1	Introduction . . . . .	111
7.2	Relationships Between CPT and FSDT . . . . .	112
7.3	Examples . . . . .	116
7.3.1	Simply Supported, Uniformly Loaded, Equilateral . . . . . Triangular Plate . . . . .	116
7.3.2	Simply Supported, Uniformly Loaded, Rectangular . . . . . Plate . . . . .	117
7.4	Relationships Between CPT and TSDT . . . . .	118
7.4.1	Introduction . . . . .	118
7.4.2	Governing Equations . . . . .	119
7.4.3	The Kirchhoff Plate Theory (CPT) . . . . .	123
7.4.4	Relationships Between the Theories . . . . .	124
7.4.5	An Accurate Simplified Relationship . . . . .	125
7.4.6	An Example . . . . .	127
7.5	Closure . . . . .	128
	Problems . . . . .	129
<b>8</b>	<b>Bending Relationships for Lévy Solutions . . . . .</b>	<b>133</b>
8.1	Introduction . . . . .	133
8.2	Governing Equations . . . . .	134
8.2.1	Introduction . . . . .	134
8.2.2	Stress Resultant-Displacement Relations . . . . .	135
8.2.3	Equilibrium Equations . . . . .	137
8.3	Bending Relationships . . . . .	137
8.3.1	General Relationships . . . . .	137
8.3.2	SSSS Plates . . . . .	143
8.3.3	SCSC Plates . . . . .	144
8.3.4	SFSF Plates . . . . .	145
8.3.5	SCSS Plates . . . . .	147
8.3.6	SFSS Plates . . . . .	148
8.4	Numerical Results . . . . .	148
8.4.1	SCSC Plates . . . . .	149
8.4.2	SFSF Plates . . . . .	151
	Problems . . . . .	152

<b>9</b>	<b>Bending Relationships for Circular and Annular Plates</b>	<b>153</b>
9.1	Governing Equations	153
9.2	Relationships Between CPT and FSDT	156
9.2.1	General Relationships	156
9.2.2	Examples	162
9.3	Relationships Between CPT and TSDT	165
9.3.1	General Relationships	165
9.3.2	An Example	169
9.4	Closure	171
	Problems	171
<b>10</b>	<b>Bending Relationships for Sectorial Plates</b>	<b>177</b>
10.1	Introduction	177
10.2	Formulation	178
10.2.1	The Kirchhoff Plate Theory (CPT)	178
10.2.2	The Mindlin Plate Theory (FSDT)	180
10.2.3	Governing Equations	181
10.3	Exact Bending Relationships	182
10.3.1	General Relationships	182
10.3.2	SSS Sectorial Plates	185
10.3.3	SSC Sectorial Plates	186
10.3.4	SSF Sectorial Plates	186
10.4	Examples	187
10.4.1	SSS Plates	187
10.4.2	SSC Plates	188
10.4.3	SSF Plates	188
10.4.4	Numerical Results	189
10.5	Conclusions	192
	Problems	192
<b>11</b>	<b>Buckling Relationships</b>	<b>195</b>
11.1	Polygonal Plates	195
11.1.1	Governing Equations	195
11.1.2	Relationships Between CPT and FSDT	199
11.1.3	Relationships Between CPT and TSDT	200

11.2 Circular Plates . . . . .	205
11.2.1 Governing Equations . . . . .	205
11.2.2 Relationship Between CPT and FSDT . . . . .	207
11.2.3 Relationship Between CPT and TSDT . . . . .	208
11.2.4 Numerical Results . . . . .	210
11.3 Sectorial Mindlin Plates . . . . .	211
11.3.1 Governing Equations . . . . .	211
11.3.2 Buckling Load Relationship . . . . .	215
Problems . . . . .	218
<b>12 Free Vibration Relationships . . . . .</b>	<b>223</b>
12.1 Introduction . . . . .	223
12.2 Relationships Between CPT and FSDT . . . . .	226
12.2.1 General Relationship . . . . .	226
12.2.2 Numerical Results . . . . .	229
12.3 Relationships Between CPT and TSDT . . . . .	235
12.4 Concluding Remarks . . . . .	241
Problems . . . . .	241
<b>13 Relationships for Inhomogeneous Plates . . . . .</b>	<b>243</b>
13.1 Deflection Relationships for Sandwich Plates . . . . .	243
13.1.1 Introduction . . . . .	243
13.1.2 Governing Equations for Kirchhoff Plates . . . . .	244
13.1.3 Governing Equations for Sandwich Mindlin Plates . . . . .	245
13.1.4 Relationship Between Sandwich and . . . . .	
Kirchhoff Plates . . . . .	248
13.1.5 Examples . . . . .	250
13.1.6 Relationship Between Sandwich . . . . .	
and Solid Mindlin Plates . . . . .	252
13.2 Deflection Relationships for Functionally Graded . . . . .	
Circular Plates . . . . .	253
13.2.1 Introduction . . . . .	253
13.2.2 Formulation . . . . .	254
13.2.3 Relationships Between CPT and FSDT . . . . .	256
13.2.4 Relationships for Various Boundary Conditions . . . . .	260
13.2.5 Illustrative Examples . . . . .	264

13.3 Buckling Load Relationships for Sandwich .....	
Mindlin Plates .....	269
13.3.1 Governing Equations .....	269
13.3.2 Buckling Load Relationship .....	271
13.4 Free Vibration Relationships for Sandwich Plates .....	272
13.4.1 Governing Equations .....	272
13.4.2 Free Vibration Relationship .....	275
13.5 Summary .....	276
<b>References</b> .....	<b>279</b>
<b>Subject Index</b> .....	<b>293</b>





## Preface

There exist many books on the theory and analysis of beams and plates. Most of the books deal with the classical (Euler-Bernoulli/Kirchhoff) theories but few include shear deformation theories in detail. The classical beam/plate theory is not adequate in providing accurate bending, buckling, and vibration results when the thickness-to-length ratio of the beam/plate is relatively large. This is because the effect of transverse shear strains, neglected in the classical theory, becomes significant in deep beams and thick plates. In such cases, shear deformation theories provide accurate solutions compared to the classical theory.

Equations governing shear deformation theories are typically more complicated than those of the classical theory. Hence it is desirable to have exact relationships between solutions of the classical theory and shear deformation theories so that whenever classical theory solutions are available, the corresponding solutions of shear deformation theories can be readily obtained. Such relationships not only furnish benchmark solutions of shear deformation theories but also provide insight into the significance of shear deformation on the response. The relationships for beams and plates have been developed by the authors and their colleagues over the last several years. However, this valuable information is dispersed in the literature. Therefore, the goal of this monograph is to bring together these relationships for beams and plates in a single volume.

The book is divided into two parts. Following the introduction, Part 1 consists of Chapters 2 to 5 dealing with beams, and Part 2 consists of Chapters 6 to 13 covering plates. Problems are included at the end of each chapter to use, extend, and develop new relationships. The book is suitable as a reference by engineers and scientists working in industry and academia. An introductory course on mechanics of materials and elasticity should prove to be helpful but not necessary because a review of the basics is included in the relevant chapters.

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