

Energy Principles and Variational Methods in Applied Mechanics

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To

The Red, White, and Blue
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The Land of Opportunity and Freedom
with sincere gratitude and appreciation

Preface

The increasing use of numerical and computational methods in engineering and applied sciences has shed new light on the importance of energy principles and variational methods. The number of engineering courses that make use of energy principles and variational formulations and methods has also grown very rapidly in recent years. In view of the increase in the use of the variational formulations and methods (including the finite element method), there is a need to introduce the concepts of energy principles and variational methods and their use in the formulation and solution of problems of mechanics to both undergraduate and beginning graduate students. This book, an extensively revised version of the author's earlier book *Energy and Variational Methods in Applied Mechanics*, is intended for senior undergraduate students and beginning graduate students in aerospace, civil, and mechanical engineering, and applied mechanics, who have had a course in fundamental engineering subjects as well as in ordinary and partial differential equations.

The book is organized into 10 chapters and is self-contained as far as the subject matter is concerned. Chapter 1 presents a general introduction to the subject of variational principles. Chapter 2 contains a brief review of the algebra and calculus of vectors and Cartesian tensors. A review of the basic equations of linear solid continuum mechanics is included in Chapter 3. These equations are frequently referred in subsequent chapters. Much of the material presented in Chapters 1 through 3 can be assigned as a reading material, especially in a graduate class.

Chapter 4 deals with the concepts of work, energy, and the basic topics from variational calculus, including the Euler equations, fundamental lemma of calculus of variations, essential and natural boundary conditions, and minimization of functionals without and with equality constraints. Virtual work and energy principles, and energy methods of solid and structural mechanics are presented in Chapter 5. Chapter 6 is devoted to a discussion of Hamilton's principle for dynamical systems. Classical variational methods of approximation (e.g., the methods of Ritz, Galerkin, Kantorovich, etc.) are presented in Chapter 7. All of the concepts and methods presented in Chapters 4 through 7 are illustrated using bars and beams although the methods discussed in Chapter 7 are readily applicable to field problems whose differential equations resemble those of bars and beams. Chapter 8 is dedicated to applications of the energy principles and variational methods developed in earlier chapters to circular and rectangular plates. In the interest of completeness and for use as a reference for approximate solutions, exact solutions are also included. The finite element method is introduced in Chapter 9, with applications to beams and plates. Displacement finite element models of Euler–Bernoulli and Timoshenko beam theories and classical and first-order shear deformation plate theories are presented. A unified approach, more

general than that found in most solid mechanics books, is used to introduce the finite element method. As a result, the student can readily extend the method to other subject areas of solid mechanics as well as to other branches of engineering. Lastly, the mixed variational principles of Hellinger and Reissner for elasticity are derived in Chapter 10. Mixed variational formulations, including mixed finite element models of beams and plates are discussed.

Each chapter of the book contains many example problems and exercises that illustrate, test, and broaden the understanding of the topics covered. A list of references, by no means complete or up-to-date, is also provided at the end of each chapter. Answers to selective problems are included at the end of the book.

The book is suitable as a textbook for a senior undergraduate course or a first-year graduate course on energy principles and variational methods taught in aerospace, civil, and mechanical engineering, and applied mechanics departments. To gain the most from the text the student should have a senior undergraduate or first year graduate standing in engineering. Some familiarity with basic courses in differential equations, mechanics of materials, and dynamics would also be helpful.

I have benefited by the professional works, encouragement, and support of many colleagues as well as students who have taught me how to explain complicated concepts in simple terms. While it is not possible to name all of them, without their help and support it would not have been possible for me to make the modest contributions to the field of mechanics through teaching, research, and writings.

My sincere thanks are due to my teacher Professor J. T. Oden (University of Texas at Austin), for many things I learned from him and have been useful all my life. In the same vein I wish to thank Professor C. W. Bert (University of Oklahoma, Norman) for giving me the opportunity to teach and develop into what I am. I am very grateful for their mentorship, advice, and support.

Deep gratitude to my wife for her patience and support while I am occupied with the writing of this book and others, and for her smile when I say every day 'I have so much to do.'

Most books are not free of errors, especially those with many mathematical equations and numbers. I wish to thank in advance those readers who are willing to draw attention to typos and errors, using the e-mail address: *jnreddy@hotmail.com*.

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NOTE: The page numbers listed here do not match with those in the printed book.

Contents

Preface	v
1 Introduction	1
1.1 Preliminary Comments	1
1.2 The Role of Energy Methods and Variational Principles	2
1.3 Some Historical Comments	2
1.4 Present Study	4
References	5
2 Mathematical Preliminaries	9
2.1 Introduction	9
2.2 Vectors	10
2.2.1 Definition of a Vector	10
2.2.2 Scalar and Vector Products	12
2.2.3 Components of a Vector	16
2.2.4 Summation Convention	17
2.2.5 Vector Calculus	21
2.2.6 Integral Relations	28
2.3 Tensors	30
2.3.1 Second-Order Tensors	30
2.3.2 General Properties of a Dyadic	33
2.3.3 Nonion Form of a Dyadic	34
2.3.4 Eigenvectors Associated with Dyadics	37
Exercises	42
References	47
3 Review of Equations of Solid Mechanics	49
3.1 Introduction	49
3.1.1 Classification of Equations	49
3.1.2 Descriptions of Motion	50
3.2 Conservation of Linear and Angular Momenta	51
3.2.1 Equations of Motion	51
3.2.2 Symmetry of Stress Tensor	54
3.3 Kinematics of Deformation	55
3.3.1 Strain Tensor	55
3.3.2 Strain Compatibility Equations	59

3.4 Constitutive Equations	62
3.4.1 Introduction.....	62
3.4.2 Generalized Hooke's Law	63
3.4.3 Plane Stress Constitutive Relations	65
3.4.4 Thermoelastic Constitutive Relations	66
Exercises.....	69
References	77
4 Work, Energy, and Variational Calculus	79
4.1 Concepts of Work and Energy	79
4.2 Strain Energy and Complementary Strain Energy	84
4.3 Virtual Work	95
4.4 Calculus of Variations	102
4.4.1 The Variational Operator.....	102
4.4.2 Functionals	105
4.4.3 The First Variation of a Functional.....	106
4.4.4 Fundamental Lemma of Variational Calculus.....	107
4.4.5 Extremum of a Functional.....	107
4.4.6 Euler Equations.....	109
4.4.7 Natural and Essential Boundary Conditions.....	112
4.4.8 Minimization of Functionals with Equality Constraints	116
Exercises.....	121
References	128
5 Energy Principles of Structural Mechanics	131
5.1 Virtual Work Principles	131
5.1.1 Introduction	131
5.1.2 The Principle of Virtual Displacements	131
5.1.3 Unit-Dummy-Displacement Method	135
5.2 Principle of Total Potential Energy and Castigliano's Theorem I	140
5.2.1 Principle of Minimum Total Potential Energy	140
5.2.2 Castigliano's Theorem I	144
5.3 Principles of Virtual Forces and Complementary Potential Energy	149
5.4 Principle of Complementary Potential Energy and Castigliano's Theorem II	153
5.5 Betti's and Maxwell's Reciprocity Theorems	162
Exercises.....	167
References	172

6 Dynamical Systems: Hamilton's Principle	175
6.1 Introduction	175
6.2 Hamilton's Principle for Particles and Rigid Bodies	175
6.3 Hamilton's Principle for a Continuum	181
6.4 Hamilton's Principle for Constrained Systems	187
6.5 Rayleigh's Method	193
Exercises	195
References	200
7 Direct Variational Methods	201
7.1 Introduction	201
7.2 Concepts from Functional Analysis	202
7.2.1 General Introduction	202
7.2.2 Linear Vector Spaces	203
7.2.3 Normed and Inner Product Spaces	207
7.2.4 Transformations, and Linear and Bilinear Forms	212
7.2.5 Minimum of a Quadratic Functional	212
7.3 The Ritz Method	218
7.3.1 Introduction	218
7.3.2 Description of the Method	218
7.3.3 Properties of Approximation Functions	221
7.3.4 Ritz Equations for the Parameters	222
7.3.5 General Features of the Method	227
7.3.6 Examples	228
7.4 General Boundary-Value Problems	241
7.4.1 Variational Formulation	241
7.4.2 Ritz Approximation	250
7.5 Weighted-Residual Methods	255
7.5.1 Introduction	255
7.5.2 Galerkin's Method	258
7.5.3 Least-Squares Method	258
7.5.4 Collocation Method	259
7.5.5 Eigenvalue and Time-Dependent Problems	259
7.5.6 Equations for Undetermined Parameters	261
7.5.7 Examples	263
7.6 Summary	282
Exercises	283
References	291
8 Theory and Analysis of Plates	293
8.1 Introduction	293
8.1.1 General Comments	293
8.1.2 An Overview of Plate/Shell Theories	295

8.2 Classical Plate Theory	299
8.2.1 Governing Equations of Circular Plates	299
8.2.2 Analysis of Circular Plates	306
8.2.3 Governing Equations in Rectangular Coordinates	327
8.2.4 Navier Solutions of Rectangular Plates.....	335
8.2.5 Lévy Solutions of Rectangular Plates	349
8.2.6 Variational Solutions: Bending	356
8.2.7 Variational Solutions: Vibration	372
8.2.8 Variational Solutions: Buckling.....	377
8.3 Shear Deformation Plate Theory.....	389
8.3.1 Governing Equations of Circular Plates	389
8.3.2 Governing Equations in Rectangular Coordinates	391
8.3.3 Exact Solutions of Axisymmetric Circular Plates.....	394
8.3.4 Exact Solutions of Rectangular Plates	398
8.3.5 Relationships Between Bending Solutions of Classical and Shear Deformation Theories.....	404
8.3.6 Variational Solutions of Circular and Rectangular Plates ..	411
Exercises.....	415
References	420
9 The Finite Element Method	423
9.1 Introduction.....	423
9.2 Finite Element Analysis of Bars	424
9.2.1 Governing Equation.....	424
9.2.2 Representation of the Domain by Finite Elements.....	425
9.2.3 Approximation Over an Element	427
9.2.4 Weak Form	431
9.2.5 Finite Element Equations	432
9.2.6 Assembly (Connectivity) of Elements.....	435
9.2.7 Imposition of Boundary Conditions.....	437
9.2.8 Calculation of Reactions and Derivatives of Solution: Postprocessing.....	438
9.3 Finite Element Analysis of the Euler-Bernoulli Beam Theory....	443
9.3.1 Governing Equation.....	443
9.3.2 Weak Form Over an Element	444
9.3.3 Derivation of the Approximation Functions	446
9.3.4 Finite Element Model	447
9.3.5 Assembly of Element Equations	448
9.3.6 Imposition of Boundary Conditions.....	446
9.4 Finite Element Models of the Timoshenko Beam Theory	453
9.4.1 Governing Equations	453
9.4.2 Displacement Finite Element Models	453
9.4.3 Reduced Integration Element (RIE)	454
9.4.4 Consistent Interpolation Element (CIE).....	456
9.4.5 Superconvergent Element (SCE).....	458

9.5 Finite Element Models of the Classical Plate Theory	460
9.5.1 Introduction	460
9.5.2 General Formulation	461
9.5.3 Conforming and Nonconforming Plate Elements	463
9.5.4 Fully Discretized Finite Element Models	464
9.6 Finite Element Models of the First-Order Plate Theory	469
9.6.1 Governing Equations and Weak Forms	469
9.6.2 Finite Element Model	470
9.6.3 Numerical Integration	473
Exercises	483
References	488
10 Mixed Variational Formulations	493
10.1 Introduction	493
10.1.1 General Comments	493
10.1.2 Mixed Variational Principles	493
10.1.3 Extremum and Stationary Behavior of Functionals	495
10.2 Stationary Variational Principles	497
10.2.1 Minimum of the Total Potential Energy	497
10.2.2 The Hellinger-Reissner Variational Principle	499
10.2.3 The Reissner Variational Principle	503
10.3 Variational Solutions Based on Mixed Formulations	503
10.4 Mixed Finite Element Model of Beams	507
10.4.1 The Euler-Bernoulli Beam Theory	507
10.4.2 The Timoshenko Beam Theory	512
10.5 Mixed Finite Element Models of the Classical Plate Theory	516
10.5.1 Preliminary Comments	516
10.5.2 Mixed Model I	517
10.5.3 Mixed Model II	521
10.6 Closure	527
Exercises	528
References	531
Answers/Solutions to Selected Exercises	535
Subject Index	569