APPLIED FUNCTIONAL ANALYSIS AND VARIATIONAL METHODS IN ENGINEERING

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To my motherland
India

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PREFACE

An increased interest is seen in recent years in the study of functional analysis among engineers and physicists who are theoretically inclined. This is because it is now widely accepted that functional analysis is a powerful tool in the solution of mathematical problems arising from physical situations. The main motivation which led me to the writing of this book came from the following observation: most engineers and physicists, whose interest is primarily in applications and who are without special training in mathematics, face a difficult task in bringing the tools of functional analysis to bear on the questions of the existence and uniqueness of solutions of mathematical problems, and the quality of their approximation by variational methods, including the finite-element method.

This book is intended to be a simple and easy introduction to functional analysis techniques that are useful in the study of differential equations arising in engineering analysis. Since most applications in engineering do not require extensive knowledge of functional analysis, only the concepts that are necessary for an engineer to equip him/herself for his/her study are presented here. In order to make the present book as accessible as possible, I have tried to avoid difficult topics while presenting concepts that are simple and useful. A greater amount of explanation and larger number of illustrative examples than is usually found in most books on the subject are also presented. In addition, it is shown how the functional analysis tools can be put to work in the formulation as well as the solution of engineering problems by the variational methods.

Readers of this book should be familiar with calculus and linear algebra, theory of ordinary and partial differential equations, vectors and matrices, and basic courses in fluid mechanics, heat transfer and mechanics of solids.

Following the introduction, the major equations of engineering are reviewed in Chapter 2. The equations developed in this chapter are studied in the later chapters from the existence and uniqueness of solutions point of view, and from their numerical solution point of view. Most graduate students of engineering are likely to have had a course in continuum mechanics or its equivalent, and therefore can skip the chapter in their first reading and refer to it whenever the need arises during the coverage of the other chapters.

In Chapter 3, an introduction to functional analysis is presented. Concepts from linear vector spaces, normed spaces and inner product spaces are systematically developed and illustrated via examples. Most of the concepts, for example, the notion of a vector, norm, inner product, orthogonality, projection, orthonormal bases, and generalized Fourier series, are rather intuitive. They are presented as natural generalizations of the corresponding concepts from the Euclidean space. This inductive rather than deductive approach should be welcomed by engineers and physicists. The abstract Banach and Hilbert spaces are introduced late in Chapter 3. This chapter constitutes a basic prerequisite for the rest of the book. Those, especially mathematics majors, who have had a course in advanced calculus and/or analysis can go straight to Chapter 4.

Chapter 4 is devoted to the discussion of linear functionals on Hilbert spaces, Sobolev spaces, generalized solutions of boundary-value problems, the minimum of a quadratic functional and concepts from calculus of variations (such as the first variation of a functional, natural and essential boundary conditions and the Euler equations). Most of these concepts are relatively familiar to engineers.

Chapter 5 deals with the questions of existence and uniqueness of linear algebraic equations, operator equations, and variational boundary-value and eigenvalue problems. The Lax–Milgram theorem and its generalizations are presented and existence and uniqueness results are included for field problems governed by the Poisson or Laplace equation, plane elasticity, plate bending, and Stokes flow problems. Much of the material presented in Chapter 5 is new to engineers and physicists.

In Chapter 6, several classical variational methods are described and used to determine the solution of various problems in engineering. These include the Ritz method, Bubnov–Galerkin method, least squares method, Kantorovich method, and Trefftz method. While these methods are familiar to most engineers, the general description and convergence results presented in the book should aid in a greater understanding of the applicability and limitations of the methods. Considerable attention is devoted to practical aspects, such as the selection of the basis of the approximation space and the convergence and stability of the numerical schemes.

Chapter 7 is devoted to the study of the finite-element method. The Ritz as well as weighted-residual finite-element models are introduced and their application to problems in one and two dimensions is demonstrated via several model problems. Applications to problems in heat transfer, fluid mechanics and solid mechanics are included. The questions of convergence and stability of various finite element models are also addressed.

Throughout the book, numerous example problems are presented, and exercise problems are included at appropriate intervals to test and extend the understanding of the concepts covered. The book can be used both as a text book in engineering and applied mathematics and as a reference for theoretically oriented engineers and physicists, and applied mathematicians.

If the book is used for a single course, Chapters 2 and 3 should be either required as prerequisites or reviewed quickly to allow sufficient time for the coverage of the remaining chapters.

The author's writings in the area of variational methods are profoundly influenced by the works of S. G. Mikhlin and K. Rectorys, among few others. The author expresses his sincere thanks to all those who have by their work, advice and support contributed to the writing of this book. Special thanks to K. Chandrashekhara, Paul Heylinger, and C. F. Liu for the proof reading during the preparation and production of the book. It is with great pleasure and appreciation the author acknowledges the patience in the skilful typing of the manuscript by Vanessa McCoy.

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