
**AN INTRODUCTION
TO THE
FINITE ELEMENT METHOD**

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An Introduction to the Finite Element Method

Third Edition

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To
My teachers and students

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PREFACE

The third edition of the book, like the previous two editions, represents an effort to select and present certain aspects of the finite element method that are most useful in developing and analyzing linear problems of engineering and science. In revising the book, students taking courses that might use this book as their textbook have been kept in mind. This edition is prepared to bring more clarity to the concepts being discussed while maintaining the necessary mathematical rigor and providing physical interpretations and engineering applications at every step.

This edition of the book is a revision of the second edition, which was well received by engineers as well as researchers in the fields of engineering and science. Most of the revisions in the current edition took place in Chapters 1 through 6 and 14. Chapter 5 on error analysis from the second edition is eliminated and a section on the same subject is added to Chapter 14. Chapter 3 from the second edition is now divided into two chapters, Chapter 3 on theoretical formulation and Chapter 4 on applications. Chapter 4 on beams from the second edition now became Chapter 5 in the current edition, making the total number of chapters in both editions the same. Another change is the interchanging of Chapters 10 and 11 to facilitate the natural transition from plane elasticity to plate bending (Chapter 12). New material on three-dimensional finite element formulations and nonlinear formulations is added in Chapter 14. In all the chapters, material is added and reorganized to aid the reader in understanding the concepts.

Another change in this edition is the removal of appendices containing the program listing of **FEM1D** and **FEM2D**, which would have taken up over 80 pages. The listings as well as the executables are available through a McGraw-Hill website for this book. The source files of these programs may be obtained from the author for a small fee.

Most people who have used the earlier editions of the book liked the “differential equations approach” adopted here. This is natural because everyone, engineer or scientist, is looking for a way to solve differential equations arising in the study of physical phenomena. It is hoped that the third edition of the book serves the student even better than the previous editions in understanding the finite element method as applied to linear problems of engineering and science.

The author has benefited by teaching an introductory course from this book for many years. While it is not possible to name hundreds of students and colleagues who have contributed to the author’s ability to explain concepts in a clear manner, the author expresses his sincere appreciation to the following people for their helpful comments on this edition:

xvi PREFACE

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Tejashwina vadheetamasthu
(*May what we study be well studied*)