The Finite Element Method for Boundary Value Problems

Mathematics and Computations

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Contents

Preface

Since there are already many textbooks and monographs on the finite element method, it is perhaps natural to ask "why another book?" This question can be answered if one examines the published material on the subject. Broadly speaking, the books on the subject can be classified into two categories: those that present the finite element method as a study in applied mathematics and those that approach the subject using specific applications. Finite element books on linear elasticity, stress analysis, heat transfer, fluid mechanics, and so on are examples of application based approach. Both types of writings have their own strengths and weaknesses from the point of view of the students wanting to learn the subject. The applied mathematics approach requires more rigorous mathematical background and preparation and as a consequence graduate students in engineering and sciences shy away from learning the subject through this approach. Secondly, these writings often lack application aspects of the subject that are generally helpful for engineering and science students. The writings that are highly focused in presenting the subject through specific applications obviously result in loss of generality and as a consequence the students often learn the subject as a technique for a specific class of problems. For example, a finite element book on linear elasticity may generally focus on minimization of total potential energy and as a consequence the students may never realize the much broader impact of the subject on all BVPs in the other areas of mechanics and applied sciences. Distinct demarcation in writings and teaching of the subject for linear processes, non-linear processes, solids, liquids and gasses often leaves the students confused and unclear not only regarding the mathematical foundations of the subject, but also its much broader impact in applications in all areas of engineering and sciences.

This book is intended to bridge the gap between the applied mathematics and strictly application-oriented books. The material in this book is presented in a mathematically rigorous fashion but with sufficient examples, applications, and illustrations in various areas of engineering, sciences, and mathematical physics so that students are able to grasp the mathematical foundation of the subject as well as its versatility of applications in all areas of engineering, sciences, and mathematical physics. The book is aimed for a first semester graduate study of the finite element method for boundary values problems (BVPs). The finite element method is introduced and presented as a method of approximation for obtaining numerical solutions of differential and partial differential equations describing time-independent processes (BVPs) regardless of their origin or field of application. In order to address the totality of all BVPs rigorously and in an application-independent fashion, the differential operators appearing in all BVPs are classified mathematically into three categories: self-adjoint, non-self adjoint, and non-linear operators, and their properties are established. These are then utilized with various methods of approximation such as the Galerkin method, Petrov-Galerkin method, weighted residual method, Galerkin method with weak form, least squares processes, and other methods from which the details of the finite element processes are derived. A correspondence is established between the methods of approximation and hence finite element method and the elements of the calculus of variations. This is then utilized for various methods of approximation for the three classes of differential operators to determine which methods of approximation yield unconditionally stable computational processes.

Chapter 1 provides a brief introduction of the subject of the mathematics of computations and the finite element method for boundary value problems. Concepts of discretization, local approximations, integral forms, element algebraic equations, assembly of element equations, computations of solutions, and post-processing of solutions are introduced. An introduction to the k-version of the finite element method and hpk framework for computations of the solutions of the boundary value problems is also presented. Basic elements from applied mathematics: spaces, scalar product spaces, scalar product and its significance, function spaces, differential operators and their mathematical classifications, and energy product are presented in chapter 2. Chapter 2 also contains elements of calculus of variations and functional analysis: concept of variation of a functional, Euler's equations, correspondence between extrema of functionals and solutions of boundary value problems, fundamental and other lemmas in calculus of variations and their proofs, Riemann and Lebesgue integrals, properties of self adjoint, non-self adjoint, and non-linear differential operators including examples. Concepts and definitions of variationally consistent (VC) and variationally inconsistent (VIC) integral forms are introduced to establish when the integral forms yield unconditionally stable computational processes.

Chapter 3 contains classical methods of approximation based on fundamental lemma such as the Galerkin method, Galerkin method with weak form, Petrov-Galerkin method, and method of weighted residuals (GM, GM/WF, PGM, WRM), and least squares method based on residual functional (LSM) for all three classes of differential operators. Many theorems and their proofs related to VC and VIC integral forms from these methods are presented for the three classes of differential operators. Applications and model problems are considered to illustrate various concepts. Serious shortcomings of classical methods of approximation for practical applications are discussed. Chapter 4 introduces the finite element model details for GM, GM/WF, PGM, WRM, and LSM for all three classes of differential operators. Specific details and formulations for model problems in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, proofs of VC and VIC integral forms for the model problems, C^0 solutions and solutions of higher classes in *hpk* framework for finite element formulations and processes for self adjoint, non-self adjoint, and non-linear differential operators are presented in chapters 5, 6, and 7, respectively.

Chapter 8 contains basic elements of mapping and interpolation theory. Details of mapping of points, lengths, areas, and volumes are discussed. Local approximations are presented for elements in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ using Lagrange, Legendre, and Chebyshev polynomials. Local approximations of class C^0 and higher classes, p-version hierarchical local approximations of class C^0 and higher classes are presented in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. Area and volume coordinates are introduced and utilized for triangular and tetrahedral family of elements in $\mathbb{R}^2$ and $\mathbb{R}^3$ to derive local approximations of various classes.

Chapter 9 presents finite element formulations in linear solid and structural mechanics, derived using principle of minimum total potential energy, formulated directly from the physics of deformation without utilizing the underlying differential equations. A general derivation applicable to finite element formulations in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ is presented first and then followed by specific examples in $\mathbb{R}^2$ such as plane stress. Chapter 10 contains derivations of finite element formulations using principle of virtual work. This approach is specially meritorious for finite deformation and finite strain reversible processes. A general derivation in $\mathbb{R}^3$ is presented first that is specialized for applications in $\mathbb{R}^1$ and $\mathbb{R}^2$. Some additional finite element formulations related to axial deformation of rods (or spars) in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, use of element local coordinate systems, and finite element formulations for Euler–Bernoulli and Timoshenko beams are presented in chapter 11. Appendix A contains details related to Gauss quadrature in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. The computer program, *Finesse* (“Finite element system”), used to solve the problems in this book is available free of cost from the first author upon request.

The material in this book is self-contained and requires no supplementary reading or any other reference material. The students learning the subject through this book are expected to have two semesters of calculus, an undergraduate course in differential and partial differential equations, a course in linear algebra and an undergraduate level course in numerical methods. An advanced course in partial differential equations and a course in calculus of variations are helpful but are not prerequisite to learning the material presented in this book. This book is a result of the evolution of the first author’s thirty years of teaching and research of the subject in the Department of Mechanical Engineering at the University of Kansas. The author’s own research work in mathematics of computations and the finite element subject and continuum mechanics has contributed heavily to this unique approach of presenting the mathematical details of the foundation of the finite element method with simplicity while maintaining its versatility and transparency for applications. The first author has successfully utilized this material in educating graduate students on the subject as well as preparing them for post-graduate studies and research.

Both authors over the last twenty years have been engaged in joint research grants, publications and collaborative research efforts that have resulted in many new concepts such as operator classifications, the *k-version of finite element method, variationally consistent integral forms*, and so on that form the foundation of much of the material in this book. Authors' long friendship and research collaborations have been extremely enjoyable and fruitful in bringing focus, depth, clarity, and in developing this unique approach of presenting the mathematics and computations related to the finite element method for boundary value problems presented in this book. The DEPSCoR/AFOSR grant to the first author and the joint research grants to the authors from the U.S. Army (ARO) related to k-version of finite element method, operator classifications, VC and VIC integral forms, and unconditionally stable computational processes resulted in a significant number of joint fundamental publications that form the core of the material in many chapters of this book. The authors are truly grateful to many of their graduate students whose Ph.D. and M.S. theses in many areas of computational mathematics and finite element method have contributed immensely in bringing the subject matter to its present level of maturity.

The first author is extremely thankful to his Ph.D. student Mr. Tyler Stone who prepared the first draft of the manuscript of this book single handedly, performed many numerical studies contained in the book, and also helped in many subsequent versions. Thanks are also due to Dr. Daniel Nunez, a former Ph.D. student of the first author, who encouraged and supported such writing endeavors and contributed heavily in many portions of chapters 6, 7, and 8 including numerical studies for model problems contained in these chapters. A very special thanks to Mr. Aaron D. Joy, the first author's current Ph.D. student who has typeset the book and has typed and retyped many portions of the book, reorganized and in many cases redid the graphs and illustrations to bring the manuscript of the book to its present level. His interest and knowledge of the subject, hard work, and commitment to this book project have been instrumental in the completion of the book. This book would not have been possible without the research grant from DEPSCoR/AFOSR to the first author and the joint research grants: W911NF-09-1-0548 (FED0065623), W-911NF-11-1-0471 (FED0061541), and W911NF-12-1-0463 from the U.S. Army Research

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This book contains so many equations, derivations, and mathematical details that it is hardly possible to avoid some typographical and other errors. Authors would be grateful to those readers who are willing to draw attention to the errors using the emails: kssurana@ku.edu or jnreddy@tamu.edu.

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ABOUT THE AUTHORS

Karan S. Surana, born in India, went to undergraduate school at Birla Institute of Technology and Science (BITS), Pilani, India, and received a B.E. degree in Mechanical Engineering in 1965. He then attended the University of Wisconsin, Madison, where he obtained M.S. and Ph.D. degrees in Mechanical Engineering in 1967 and 1970, respectively. He worked in industry, in research and development in various areas of computational mechanics and software development, for fifteen years: SDRC, Cincinnati (1970–1973), EMRC, Detroit (1973–1978); and McDonnell Douglas, St. Louis (1978–1984). In 1984, he joined the Department of Mechanical Engineering faculty at University of Kansas, where he is currently the Deane E. Ackers University Distinguished Professor of Mechanical Engineering.

His areas of interest and expertise are computational mathematics, computational mechanics, and continuum mechanics. He is author of over 350 research reports, conference papers, and journal articles. He has served as advisor and chairman of 50 M.S. students and 22 Ph.D. students in various areas of Computational Mathematics and Continuum Mechanics. He has delivered many plenary and keynote lectures in various national and international conferences and congresses on computational mathematics, computational mechanics, and continuum mechanics. He has served on international advisory committees of many conferences and has co-organized minisymposia on k-version of the finite element method, computational methods, and constitutive theories at US National Congresses of Computational Mechanics organized by the US Association of Computational Mechanics (USACM). He is a member of International Association of Computational Mechanics (IACM) and USACM, and a fellow and life member of ASME.

Dr. Surana's most notable contributions include: large deformation finite element formulations of shells, the k-version of the finite element method, operator classification and variationally consistent integral forms in methods of approximations, and ordered rate constitutive theories for solid and fluent continua. His most recent and present research work is in non-classical internal polar continuum theories and non-classical Cosserat continuum theories for solid and fluent continua and associated ordered rate constitutive theories. He is author of recently published continuum mechanics textbook, *Advanced Mechanics of Continua*, CRC/Taylor & Francis, Boca Raton, Florida, 2015.

J. N. Reddy was born in *Telangana*, India, and obtained his B,E. (Mech) from Osmania University, Hyderabad, in 1968. He obtained M.S. from Oklahoma State University (1970) and Ph.D. from University of Alabama in Huntsville (1973). He worked at Lockheed Missiles and Space Company for a short period in 1974 and served on the faculty in the School of Aerospace, Mechanical, and Nuclear Engineering at University of Oklahoma (1975–1980), in the Department of Engineering Science and Mechanics at Virginia Polytechnic Institute and State University (1980–1992). Since 1992 he has been in the Department of Mechanical Engineering at Texas A&M University, currently holding the titles University Distinguished Professor, Regents Professor, and the Oscar S. Wyatt Endowed Chair Professor. Professor Reddy is internationally-recognized for his research on mechanics of composite materials and for computational methods. The shear deformation plate and shell theories that he developed bear his name (Reddy third-order shear deformation theory and Reddy layerwise theory) in the literature. The finite element formulations and models he developed have been implemented into commercial software like ABAQUS, NISA, and HyperXtrude. He is the author of numerous journal papers and 20 textbooks, several of them with multiple editions. Dr. Reddy is one of the original top 100 ISI Highly Cited Researchers in Engineering around world with over 19,200 citations with h-index of over 66 as per Web of Science; the number of citations is over 47,000 with h-index of 89 and i10-index of 405 (i.e., 405 papers are cited at least 10 times) as per Google Scholar.

Most significant awards and honors he received to date are: *1992 Worcester Reed Warner Medal* and *1995 Charles Russ Richards Memorial Award* of the American Society of Mechanical Engineers (ASME); *1997 Archie Higdon Distinguished Educator Award* from the Mechanics Division of the American Society of Engineering Education; *1998 Nathan M. Newmark Medal* from the American Society of Civil Engineers (ASCE); *2000 Excellence in the Field of Composites* and *2004 Distinguished Research Award* from the American Society for Composites (ASC); *2003 Computational Solid Mechanics* award from the US Association of Computational Mechanics; *2014 The IACM O.C. Zienkiewicz Award* from the International Association of Computational Mechanics; *2014 Raymond D. Mindlin Medal* from the Engineering Mechanics Institute of ASCE; and *2016 William Prager Medal* of the Society of Engineering Science. He is an elected *member of the US National Academy of Engineering* for contributions to composite structures and to engineering education and practice and elected as a *Foreign Fellow of the Indian National Academy of Engineering*. He is a fellow of many professional societies (e.g., AIAA, ASC, ASCE, ASME, AAM, USACM, IACM), and serves on the editorial boards of two dozen journals. A more complete resume can be found at **http://www.tamu.edu/acml/.**