Oden
Reddy

An Introduction to the MATHEMATICAL THEORY OF FINITE ELEMENTS

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AN INTRODUCTION TO THE MATHEMATICAL THEORY OF FINITE ELEMENTS

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PREFACE

During a remarkably short span of years the subject of finite elements has expanded from a collection of effective techniques for solving practical problems in engineering and science to a rich and exciting branch of applied mathematics. The aim of this book is to present the student of engineering science or applied mathematics an introductory account of this mathematical theory.

The book has developed as a result of seminars and courses on finite-element theory taught by the authors at five universities in recent years to students with diverse backgrounds and often modest mathematical preparation. For such an audience, we have found it effective to begin the study with basic mathematical concepts and to systematically build on these the elements of approximation theory, Hilbert spaces, and partial differential equations essential to an understanding of the most important aspects of linear finite-element theory. This book essentially follows this plan. To keep the size and scope of the work within reasonable limits, it has been necessary to omit several important topics in favor of more basic ones. For example, we have not included material on nonlinear problems, integral equations, or eigenvalue problems. However, some of these subjects should be easily mastered by the reader of this book; other subjects must await study in future works.

We owe a great deal to those who developed the mathematical theory of finite elements in recent years. We have been particularly influenced by the work of Ivo Babuška and J. P. Aubin, and we have profited not only in writing this book but also in our own research, from the writings of Philippe Ciarlet, P. A. Raviart, J. L. Lions, and others, and from numerous discussions with our colleague, Ralph Showalter. The first author registers a special note of thanks to the Finite-Element Circus and to certain members of the Circus who have patiently discussed the subject with him; particularly Ivo Babuška, Jim Douglas, Ridgway Scott, Gilbert Strang, Al Schatz, Bruce Kellogg, Mary Wheeler, and James Bramble. We are also thankful for the advice we have received from several colleagues who read an early draft of the manuscript. In particular, we have benefited from the suggestions of John Cannon and Linda Hayes, who read the entire manuscript, and from the comments of Philippe Ciarlet. We also express thanks to M. G. Sheu, N. Kikuchi, and C. T. Reddy who helped with the proofreading. Much of our work on finite-element methods has been supported through the Air Force Office of Scientific Research and the U. S. National Science Foundation. We express our sincere gratitude for this support.

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