THEORY AND ANALYSIS OF ELASTIC PLATES AND SHELLS

Second Edition

Theory and Analysis of Elastic Plates and Shells

Second Edition

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"Whence all creation had its origin,
he, whether he fashioned it or whether he did not,
he, who surveys it all from highest heaven,
he knows—or maybe even he does not know."

Rig Veda

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Preface to the Second Edition

The objective of this second edition of *Theory and Analysis of Elastic Plates and Shells* remains the same – to present a complete and up-to-date treatment of classical as well as shear deformation plate and shell theories and their solutions by analytical and numerical methods. New material has been added in most chapters, along with some rearrangement of topics to improve the clarity of the overall presentation.

The first 10 chapters are the same as those in the first edition, with minor changes to the text. Section 2.3 on Castigliano's Theorems, Section 5.6 on axisymmetric buckling of circular plates, and Section 10.5 on relationships between the solutions of classical and shear deformation theories are new. Chapter 11 is entirely new and deals with theory and analysis of shells, while Chapter 12 is the same as the old Chapter 11, with the exception of a major new section on nonlinear finite element analysis of plates.

This edition of the book, like the first, is suitable as a textbook for a first course on theory and analysis of plates and shells in aerospace, civil, mechanical, and mechanics curricula. Due to the coverage of the linear and nonlinear finite element analysis, the book may be used as a reference for courses on finite element analysis. It can also be used as a reference by structural engineers and scientists working in industry and academia on plates and shell structures. An introductory course on mechanics of materials and elasticity should prove to be helpful, but not necessary, because a review of the basics is included in the first two chapters of the book.

A solutions manual is available from the publisher for those instructors who adopt the book as a textbook for a course.

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Preface

The objective of this book is to present a complete and up-to-date treatment of classical as well as shear deformation plate theories and their solutions by analytical and numerical methods. *Beams* and *plates* are common structural elements of most engineering structures, including aerospace, automotive, and civil engineering structures, and their study, both from theoretical and analysis points of view, is fundamental to the understanding of the behavior of such structures.

There exists a number of books on the theory of plates and most of them cover the classical Kirchhoff plate theory in detail and present the Navier solutions of the theory for rectangular plates. Much of the latest developments in shear deformation plate theories and their finite element models have not been compiled in a textbook form. The present book is aimed at filling this void in the literature.

The motivation that led to the writing of the present book has come from many years of the author's research in the development of shear deformation plate theories and their analysis by the finite element method, and also from the fact that there does not exist a book that contains a detailed coverage of shear deformation beam and plate theories, analytical solutions, and finite element models in one volume. The present book fulfills the need for a complete treatment of the classical and shear deformation theories of plates and their solution by analytical and numerical methods.

Some mathematical preliminaries, equations of elasticity, and virtual work principles and variational methods are reviewed in Chapters 1 and 2. A reader who has had a course in elasticity or energy and variational principles of mechanics may skip these chapters and go directly to Chapter 3, where a complete derivation of the equations of motion of the classical plate theory (CPT) is presented. Solutions for cylindrical bending, buckling, natural vibration, and transient response of plate strips are developed in Chapter 4. A detailed treatment of circular plates is undertaken in Chapter 5, and analytical and Rayleigh–Ritz solutions of axisymmetric and asymmetric bending are presented for various boundary conditions and loads. A brief discussion of natural vibrations of circular plates is also included here.

Chapter 6 is dedicated to the bending of rectangular plates with all edges simply supported, and the Navier and Rayleigh–Ritz solutions are presented. Bending of rectangular plates with general boundary conditions are treated in Chapter 7. The Lévy solutions are presented for rectangular plates with two parallel edges simply supported while the other two have arbitrary boundary conditions; the Rayleigh–Ritz solutions are presented for rectangular plates with arbitrary conditions. General buckling of rectangular plates under various boundary conditions is presented in Chapter 8. The Navier, Lévy, and Rayleigh–Ritz solutions are developed here. Chapter 9 is devoted to the dynamic analysis of rectangular plates, where solutions are developed for free vibration and transient response.

The first-order and third-order shear deformation plate theories are discussed in Chapter 10. Analytical solutions presented in these chapters are limited to rectangular plates with simply supported boundary conditions on all four edges (the Navier solution). Parametric effects of the material orthotropy and plate aspect ratio on bending deflections and stresses, buckling loads, and vibration frequencies are discussed. Finally, Chapter 11 deals with the linear finite element analysis of beams and plates. Finite element models based on both classical and first-order shear deformation plate theories are developed and numerical results are presented.

The book is suitable as a textbook for a first course on theory of plates in civil, aerospace, mechanical, and mechanics curricula. It can be used as a reference by engineers and scientists working in industry and academia. An introductory course on mechanics of materials and elasticity should prove to be helpful, but not necessary, because a review of the basics is included in the first two chapters of the book.

The author's research in the area of plates over the years has been supported through research grants from the Air Force Office of Scientific Research (AFOSR), the Army Research Office (ARO), and the Office of Naval Research (ONR). The support is gratefully acknowledged. The author also wishes to express his appreciation to Dr. Filis T. Kokkinos for his help with the illustrations in this book.

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